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MULTIPLE SOLUTIONS OF THE FOURTH-ORDER EMDEN-FOWLER EQUATION

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Abstract. Two-point problem for the fourth-order Emden-Fowler equation is considered. If the given equation can be reduced to a quasi-linear one with a non-resonant linear part so that both equations are equivalent in some domain D, and if solution of the quasi-linear problem is located in D, then the original problem has a solution. We show that a quasi-linear problem has a solution of definite type which corresponds to the type of the linear part. If quasilinearization is possible for essentially different linear parts, then the original problem has multiple solutions.

Key words: quasi-linear equation, quasilinearization, conjugate point, i-nonresonant linear part, i-type solution

1. Introduction

Consider the nonlinear differential equation

$$x^{(4)} = f(t, x), \quad t \in I := [0, 1], \tag{1.1}$$

with the boundary conditions

$$x(0) = x'(0) = 0 = x(1) = x'(1). \tag{1.2}$$

Function $f : I \times \mathbb{R} \to \mathbb{R}$ is supposed to be continuous together with the partial derivative f_x. Then the unique solvability of the Cauchy problem

$$x(0) = x_0, \;\; x'(0) = x_1, \;\; x''(0) = x_2, \;\; x'''(0) = x_3$$

is ensured as well as the continuous dependence of solutions on initial data. Our research is motivated by the papers of R. Conti [1] and L. Erbe [2], who studied oscillatory properties of solutions of two-point boundary value problems.

Consider also the quasi-linear equation

$$\big(L_4 x\big)(t) := x^{(4)} - k^4 x = F(t,x), \tag{1.3}$$

where $F, F_x, : I \times \mathbb{R} \to \mathbb{R}$ are continuous and F is bounded, that is, there exists $M \in (0, +\infty)$ such that $|F(t,x,)| < M \quad \forall (t,x) \in I \times \mathbb{R}$. If the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is non-resonant with respect to the given boundary conditions (1.2), that is, the homogeneous problem $(L_4 x)(t) = 0$, (1.2) has only the trivial solution, then problem (1.3), (1.2) is solvable. Suppose that equations (1.1) and (1.3) are equivalent in a domain

$$D(t,x) = \{(t,x): \quad 0 \le t \le 1, \quad |x| \le N\}.$$

If any solution $x(t)$ of the problem (1.3), (1.2) is located in this domain of equivalence $D(t,x)$, in the other words, if the solution $x(t)$ satisfies the estimate

$$|x(t)| \le N \quad \forall t \in I, \tag{1.4}$$

then it solves also the problem (1.1), (1.2). We will say for brevity that the problem (1.1), (1.2) allows for quasilinearization with respect to the linear part $(L_4 x)(t)$.

If the equation (1.1) can be reduced to another quasi-linear equation

$$\big(l_4 x\big)(t) = F_1(t,\, x), \tag{1.5}$$

which is equivalent to (1.1) in another domain $D_1(t,x)$, then the original problem (1.1), (1.2) in some cases has another solution $x_1(t) \in D_1(t,x)$. In this way one can obtain the multiplicity results [4].

2. Quasi-Linear Problems and Types of Solution

We first prove results for quasi-linear problems of the type (1.3), (1.2) if the following condition is satisfied for any $(t,\, x)$

$$k^4 + \frac{\partial F}{\partial x}(t,\, x) > 0. \tag{2.1}$$

In our investigation we use the oscillation theory by Leighton-Nehari for the fourth-order linear differential equations [3]. We use their definition of a conjugate point.

Definition 1. A point η is called a *conjugate point* for the point $t = 0$, if there exists a nontrivial solution $x(t)$ such that

$$x(0) = x'(0) = 0 = x(\eta) = x'(\eta).$$

The conjugate points (or double zeros) in the oscillation theory for the fourth-order linear differential equations play the same role as the ordinary zeros in the oscillation theory for the second-order equations.

We define i-nonresonanse of the linear part and an i-type solution similarly as for the second-order quasi-linear problems [4, 5].

Definition 2. We will say that the linear part

$$(L_4x)(t) := x^{(4)} - k^4x$$

is i-nonresonant with respect to the boundary conditions (1.2), if there are exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point.

Definition 3. We will say that $\xi(t)$ is an i-type solution of the problem (1.3), (1.2), if for small enough $\alpha, \beta > 0$ the difference $u(t; \alpha, \beta) = x(t; \alpha, \beta) - \xi(t)$ has exactly i double zeros (or conjugate points) in the interval $(0, 1)$ and $u(1; \alpha, \beta) \neq 0$, where $x(t; \alpha, \beta)$ is a solution of (1.3), which satisfies the initial conditions

$$x(0; \alpha, \beta) = \xi(0), \quad x'(0; \alpha, \beta) = \xi'(0), \tag{2.2}$$

$$x''(0; \alpha, \beta) = \xi''(0) + \alpha, \quad x'''(0; \alpha, \beta) = \xi'''(0) - \beta. \tag{2.3}$$

In what follows we call the solution $x(t; \alpha, \beta)$ by *neighbouring* solution.

Remark 1. An i-type solution $\xi(t)$ of the problem (1.3), (1.2) has the following characteristics in terms of the variational equation: if a linear equation of variations

$$y^{(4)} - k^4y = F_x(t, \xi(t))y$$

has exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point, then $\xi(t)$ is an i-type solution. However, if $t = 1$ is a conjugate point, then $\xi(t)$ may be an i-type solution, or it may be an $(i + 1)$-type solution, or its type may be indefinite. The respective examples can be constructed.

The following theorem is valid.

Theorem 1. *The quasi-linear problem* (1.3), (1.2) *has an i-type solution, if the condition* (2.1) *is fulfilled and the linear part* $(L_4x)(t) = x^{(4)} - k^4x$ *is i-nonresonant.*

3. Emden-Fowler Equation

We apply Theorem 1 to the problem

$$x^{(4)} = \lambda^2 \, |x|^p \operatorname{sign} x, \tag{3.1}$$

where $\lambda \neq 0, \quad p > 0, \quad p \neq 1$, with the boundary conditions (1.2).

First we consider the linear equation

$$x^{(4)} - k^4x = 0, \tag{3.2}$$

where k satisfies the non-resonance condition $\cos k \cosh k \neq 1$.

Suppose that there exist n conjugate points for equation (3.2). We can construct the Green's function $G_k(t,\,s)$ for the problem (3.2), (1.2). Let denote by $\varGamma_k$ some number greater than $\sup\limits_{t,s\in[0,\,1]}\bigl|G_k(t,\,s)\bigr|$. Choose $N_k>0$ and consider the corresponding quasi-linear equation

$$x^{(4)}-k^4x=\varphi(x)\,\{\lambda^2\,|x|^p\,\mathrm{sign}\,x-k^4x\}=:f_k(x), \tag{3.3}$$

where $\varphi=1$, if $|x(t)|\le N_k$ and $f_k(x)$ (that is the right side in (3.3)) is smooth and bounded by $M_k>0$.

Quasi-linear problem (3.3), (1.2) can be written in the integral form

$$x(t)=\int\limits_0^1 G_k(t,s)f_k(x(s))\,ds,$$

from which it follows that

$$\bigl|x(t)\bigr|\le \varGamma_k\,M_k.$$

If moreover the inequality

$$\varGamma_k\,M_k\;<\;N_k \tag{3.4}$$

holds, then equations (3.1) and (3.3) are equivalent in the domain

$$\varOmega_k=\{(t,x):\;\;0\le t\le 1,\;|x|<N_k\}.$$

By Theorem 1 , the problem (3.3), (1.2) has a solution of n-type $x_n(t)$. Since, by (3.4)

$$\bigl|x_n(t)\bigr|\;<\;N_k\quad\forall t\in I$$

this solution $x_n(t)$ solves also the original problem (3.1), (1.2). If this procedure can be applied multiply (with essentially different linear parts), then the problem (3.1), (1.2) is shown to have multiple solutions.

4. Green's Function

As a by-product, we have constructed the Green's function for the oscillatory fourth-order linear problem

$$\begin{cases} x^{(4)}-k^4x=0,\\ x(0)=x'(0)=0=x(1)=x'(1)\end{cases} \tag{4.1}$$

and we give the formula and the estimate below.

Proposition 1. *The Green's function of the problem* (4.1)*, is given by*

$$G_k(t,\,s)=\begin{cases}\dfrac{1}{\varDelta}\Bigl(-u^*(t,s)\cdot v(1)-u(1)\cdot v^*(t,s)+\sum\limits_{\tau=s,\,t}\bigl[u(\tau)\cdot v(t+s-\tau)\\ -u(\tau-1)\cdot v(t+s-1-\tau)-u(t-\tau)\cdot v(\tau-s)\bigr]\Bigr),0\le s\le t\le 1,\\ \dfrac{1}{\varDelta}\Bigl(-u^*(s,\,t)\cdot v(1)-u(1)\cdot v^*(s,t)+\sum\limits_{\tau=s,\,t}\bigl[u(\tau)\cdot v(t+s-\tau)\\ -u(\tau-1)\cdot v(t+s-1-\tau)+u(t-\tau)\cdot v(\tau-s)\bigr]\Bigr),0\le t<s\le 1,\end{cases}$$

where $\Delta = 4k^3(\cos k \cosh k - 1)$ *and* u, v *are vector-functions such, that*

$$\begin{aligned}
&u(\tau) = [-\sin k\tau,\, \cos k\tau], \quad v(\tau) = [\cosh k\tau,\, \sinh k\tau],\\
&u^*(t,\, s) = [-\sin k(s - t + 1),\, \cos k(t + s - 1)],\\
&v^*(t,\, s) = [\cosh\ k(t + s - 1),\, \sinh k(s - t + 1)],
\end{aligned}$$

and the $u \cdot v$ *denotes the scalar product as usually.*

Proposition 2. *The Green's function* $G_k(t, s)$ *can be estimated by*

$$\big|G_k(t,s)\big| \le \frac{(5+\sqrt{2})\sqrt{\cosh 2k} + \sinh k + 1}{4k^3\big|\cos k \cosh k - 1\big|} =: \Gamma_k. \tag{4.2}$$

We can improve this estimate for some numbers k. If $k = \pi n$, $n \ge 1$, Green's function $G_k(t,\, s)$ can be simplified and we obtain the following estimates

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})e^k}{k^3(e^k+1)} =: \Gamma_1(k), \quad \text{if } k = (2n-1)\pi, \tag{4.3}$$

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})e^k}{k^3(e^k-1)} =: \Gamma_2(k), \quad \text{if } k = 2n\pi. \tag{4.4}$$

5. Applications

Let us return to the problem under consideration

$$\begin{cases} x^{(4)} = \lambda^2 |x|^p \operatorname{sign} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1), \end{cases}$$

or

$$\begin{cases} x^{(4)} - k^4 x = \varphi(x)\{\lambda^2 |x|^p \operatorname{sign} x - k^4 x\} =: f_k(x), \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{cases}$$

We can choose $N_k = \left(\dfrac{k^4}{\lambda^2}\right)^{\frac{1}{p-1}} \beta$, where β is a unique positive root of the equation

$$\beta^p = \beta + (p-1)p^{\frac{p}{1-p}}.$$

Then M_k, which bounds $f_k(x)$, is an absolute value of this function at the point of extremum. It can be calculated

$$M_k = \lambda^{\frac{2}{1-p}} \left(\frac{k^4}{p}\right)^{\frac{p}{p-1}} |p-1|.$$

So the inequality $\Gamma_k\, M_k < N_k$ turns to

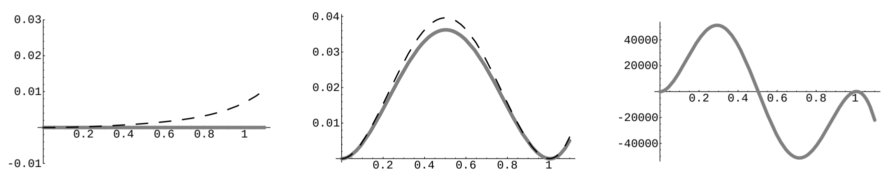
$$k\,\frac{(1+\sqrt{2})e^k}{(e^k+1)} < \beta\, \frac{p^{\frac{p}{p-1}}}{|p-1|}, \quad \text{for } k = (2n-1)\pi, \tag{5.1}$$

$$k\,\frac{(1+\sqrt{2})e^k}{(e^k-1)} < \beta\,\frac{p^{\frac{p}{p-1}}}{|p-1|}, \quad \text{for } \ k = 2n\pi. \tag{5.2}$$

We have obtained the results of calculations . They are shown that certain values of k in the form $k = \pi n,\ n = 1, 2 \ldots$ are good for the inequalities (5.1) and (5.2) to be satisfied. For instance, if $p = \frac{8}{7}$, then there exist three values of k, which satisfy the inequalities above, it means that there exist at least three solutions of the different types.

We have constructed the different solutions for the Emden-Fowler equation

$$\begin{cases} x^{(4)} = 810 \cdot |x|^{\frac{8}{7}} \operatorname{sign} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{cases} \tag{5.3}$$



a b c

Figure 1. a - 0-type solution, b - 1-type solution, c - 2-type solution of the problem (5.3).

The solid line in Fig. 1 indicates a solution of the problem (5.3) and dashed line – the corresponding neighbouring solution (see Definition 3). The solutions of the different types have different oscillatory properties and initial data.

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