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WENO SCHEMES FOR FINITE VOLUME SIMULATION OF COMPRESSIBLE FLOWS

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Abstract. This paper paper discusses accuracy of WENO reconstruction used for unstructured grids and applied to two common discretization approaches within Finite Volume Method (FVM), Cell Centered and Vertex Centered methods. The numerical results are shown for 3D transonic flow around ONERA M6 wing.
Key words: WENO, high–order methods, transonic flow

1. Introduction

Effective simulation of compressible (transonic) flows around complex 3D geometries remains a difficult task. This is especially so, when accurate predictions of integral coefficients (drag and lift) are required. In particular the drag coefficient is very sensitive to spurious entropy generated by the numerical scheme. As a result the numerical discretization has to fulfill a series of conditions:

- full conservation of mass, momentum and total energy,
- high–order of accuracy in the regions where the solution is continuous,
- oscillation-free behaviour in the vicinity of shock–waves and slip–lines.

The latter two conditions remain in a direct conflict (due to the Godunov barrier) if linear discretization scheme is used.

For the Finite Volume method the conservation property is automatically fulfilled. In its Cell-Centered version the unknowns are located at the cell centers. In order to calculate fluxes at the cell walls, the solution is reconstructed within each cell using the information from its direct neighbourhood. In the Vertex Centered version the unknowns are located at the mesh nodes. In order to avoid oscillations on discontinuities, WENO (*Weighted Essentially Non Oscillatory*) schemes are used. This approach introduced in [1, 2, 3, 4, 5, 6] is suitable for general unstructured/hybrid grids and overcomes limitations typical for classical MUSCLE schemes.

2. Finite Volume Scheme

The Euler model of fluid is used in the present paper. The equations in conservative form can be expressed as:

$$\frac{\partial U}{\partial t} + \nabla \boldsymbol{\mathcal{F}}(U) = 0, \tag{2.1}$$

where

$$U = \begin{bmatrix} \rho \\ \rho \boldsymbol{v} \\ \rho E \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} \rho \boldsymbol{v}^T \\ \rho \boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{I} p \\ \rho h \boldsymbol{v}^T \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

ρ is the density, $\boldsymbol{v}$ the velocity vector, E energy, p pressure, h enthalpy.

Integrating (2.1) over the control volume Ω_h one obtains:

$$\frac{d}{dt} \int_{\Omega_h} U d\Omega = - \int_{\partial \Omega_h} \boldsymbol{\mathcal{F}}(U) \cdot \mathbf{n} \; ds. \tag{2.2}$$

Taking further $\tilde{U}$ as cell average value of U Eq. (2.2) can be rewritten as:

$$\frac{d}{dt} \tilde{U} = - \frac{1}{|\Omega_h|} \sum_j \boldsymbol{\mathcal{F}}_j^\star, \tag{2.3}$$

where $\boldsymbol{\mathcal{F}}_j^\star$ is the numerical flux which must be calculated using U at each cell face rather than the averaged value taken from the cell center.

Collection of equations (2.3) written for all control volumes forms a system of nonlinear differential equations. In the present paper only a stationary case is considered. In the cell–centered version control volumes coincide with mesh cells. In the vertex–centered version control volumes are built around mesh nodes (similarly as dual cells are built in the Voronoi diagram).

3. Weighted Essentially Non–Oscillatory Schemes

The calculation of $\boldsymbol{\mathcal{F}}_j^\star$ relies on values at the boundary of the control volume. Since only average values $\tilde{U}$ are known, it is necessary to reconstruct the function $U(x)$ inside the control volume.

The linear reconstruction function valid inside the i-th control volume can be expressed ($\boldsymbol{x}_i$ denotes center of gravity of i-th cell) as:

$$U(\boldsymbol{x}) = \tilde{U}_i + [\nabla U]_i \cdot (\boldsymbol{x} - \boldsymbol{x}_i), \tag{3.1}$$

where gradient $[\nabla U]_i$ has to be evaluated using the average function values from the neighbouring cells. The main difficulty is caused by the fact that the solution contains discontinuities (e.g., shockwaves, slip lines). Standard approaches (like central schemes) lead to oscillations in the vicinity of the discontinuity, therefore special treatment is necessary.

In contrast reconstruction based on the WENO scheme weights gradients obtained from neighbouring stencils in order to continuously eliminate these which cause oscillations.

3.1. Multidimensional WENO reconstruction

A typical WENO finite volume scheme can be constructed as follows (for scalar function P):

- Identify several stencils $S_1, S_2, S_3 \cdots, S_m$ for a control volume Ω_h consisting of the neighbouring control volumes, such that the control volume Ω_h belongs to each stencil.
- Obtain a lower order reconstruction polynomial, denoted by P_i and associated with stencil S_i, which approximates solution on Ω_h. This point is modified for Extended WENO reconstruction.
- Calculate oscillation indicator o_i for each function P_i.
- Calculate weights for each P_i using oscillation indicator o_i.
- Find global reconstruction function for a control volume Ω_h as a weighted average of all P_i.

3.2. Calculation of nonlinear weights based on the smoothness measurement

The reconstruction function for each control volume is defined as a weighted average:

$$P = \sum_{i=1}^{m} \omega_i \cdot P_i.$$

Oscillation indicator can be defined as:

$$o_i = \| \nabla P \|_2,$$

while the weights are calculated using algebraic formula

$$\omega_i = \frac{(\epsilon + o_i)^{-r}}{\sum_{i=1}^{M} (\epsilon + o_i)^{-r}}. \tag{3.2}$$

3.3. First order method for gradient reconstruction

We shall assume now that φ_p denotes function value at the p-th cell (p=0 corresponds to the cell in which gradient is reconstructed). The Taylor expansion formula can be written as:

$$\varphi_p - \varphi_0 = \overrightarrow{r}_{p0}^{T} \nabla\varphi|_0 + \frac{1}{2} \overrightarrow{r}_{p0}^{T} \nabla^2\varphi|_0 \overrightarrow{r}_{p0} + O\left(h^3\right), \tag{3.3}$$

where $\overrightarrow{r}_{p0} = \overrightarrow{r}_p - \overrightarrow{r}_0$ denotes vector joining respective cell centres and h denotes the characteristic distance between centres of neighbouring cells.
Gradient will be approximated as a linear combination of the valuable function φ increments:

$$\overrightarrow{G}(\varphi) = \sum_{p=1}^{m} \overrightarrow{G}_p w_p (\varphi_p - \varphi_0) + O\left(h^2\right), \tag{3.4}$$

where w_p are the known weights (not to be confused with WENO weights (3.2)) and m denotes number of available stencils (m=4 on 3D tetrahedral meshes). Usually w_p is taken equal to $\|\overrightarrow{r}_{p0}\|^{-1}$, so that the coefficients $\overrightarrow{G}_p$ are "dimensionless".

The unknown coefficients $\overrightarrow{G}_p$, depend only on the local geometry of the mesh. In order to find $\overrightarrow{G}_p$ (3.4) is substituted into (3.3)

$$\overrightarrow{G}(\varphi) = \left[\sum_{p=1}^{m} \overrightarrow{G}_p w_p \overrightarrow{r}_{p0}^{T}\right] \nabla\varphi|_0 + \sum_{p=1}^{m} \overrightarrow{G}_p w_p \overrightarrow{r}_{p0}^{T} \nabla^2\varphi|_0 \overrightarrow{r}_{p0} + \dots \qquad (3.5)$$

In the formula above $\overrightarrow{G}_p$ represents gradient only if the expression in brackets forms an identity matrix (the second term describes the error of the formula). Finding $\overrightarrow{G}_p$ requires solving a number of local under/over–determined linear systems. This is done by applying a sequence of Householder transformations. This requirement can be expressed in general as:

$$\forall \overrightarrow{\nu} \in R^N \qquad \sum_{p=1}^{m} \overrightarrow{G}_p w_p \overrightarrow{r}_{p0}^{T} \overrightarrow{\nu} = \overrightarrow{\nu}, \qquad (3.6)$$

or using the basis $\left\{\overrightarrow{e^1}, \overrightarrow{e^2}, \dots, \overrightarrow{e^N}\right\}$ in $\mathbb{R}^N$ one obtains a system of linear equations:

$$\sum_{p=1}^{m} \overrightarrow{G}_p w_p \overrightarrow{r}_{p0}^{T} \overrightarrow{e}^s = \overrightarrow{e}^s, \qquad s = 1, 2, \dots, N,$$

with $m\,N$ scalar unknowns and N^2 equations. The number of available stencils m is larger than the space dimension, therefore the system is solved in the least–square sense which minimise the L_2 norm of the obtained solution. This explains why weights introduced in (3.4) are indeed necessary.

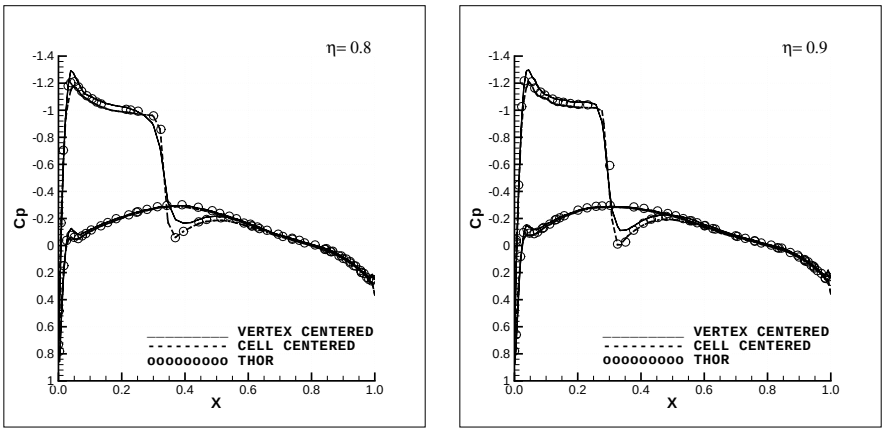
4. Numerical Results

4.1. Transonic flow past M6 ONERA wing

The test case used for comparison of CC and VC FVM is the flow past M6 ONERA wing with Mach number at infinity equal to 0.84. For such a flow two oblique shock waves merge near the tip of the wing on the upper surface.

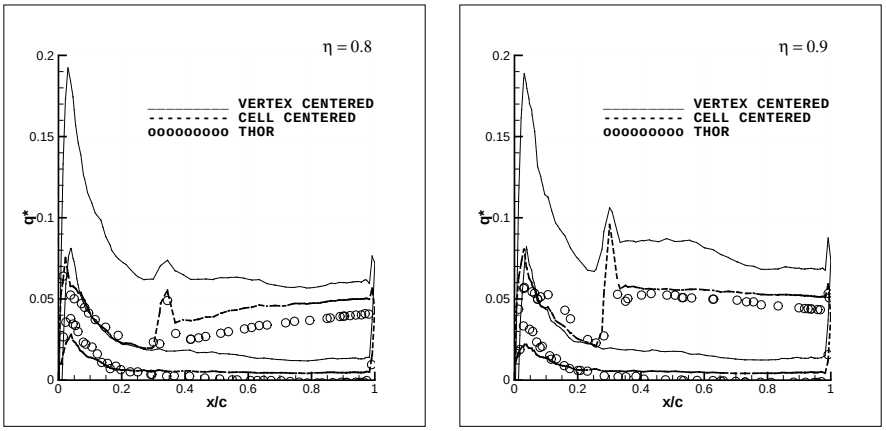
Additional results obtained by the code THOR based on the Residual Distributions Schemes [1] and developed in Von Karman Institute [5], were used as the reference for quantitative comparison. All calculations were performed on the same grid which consisted of 316275 nodes, 1940182 cells, 2289199 edges and 3913107 faces.

The comparison of Figures 1 and 2 shows that solution obtained by VC FVM method remains much more dissipative. The shock waves are sharper



a) b)

Figure 1. Pressure coefficient for different sections of M6 ONERA wing (η denotes the relative position of the cross section), Ma=0.84, α=3.04.



a) b)

Figure 2. Total pressure loss for different sections of M6 ONERA wing (η denotes the relative position of the cross section), Ma=0.84, α=3.04.

and more distinct in the CC FVM solution. It is also visible in the quantitative comparison of the c_p coefficient (Fig. 1) and of the total pressure loss (Fig. 2). The latter quantity corresponds directly to the error of the method. According to theory, total pressure should be constant everywhere with exception of shockwaves where it should increase. The parameter q^* presented in Fig. 1 is defined as:

$$q^* = \frac{q}{q_\infty} - 1,$$

where q and q_∞ denote total pressure at a given point and at infinity, respectively.

5. Conclusions

- FVM with WENO reconstruction gives for both versions of the method (CC and VC) results which are qualitatively correct. The solution near shock–waves remains oscillation-free. Shock waves remain smeared over 1-3 cells only.
- Cell Centered discretization provides better resolution of shockwaves and other flow features, than its Vertex Centered counterpart. It provides also better overall accuracy and generates significantly smaller amount of spurious entropy. For the transonic flow around ONERA M6 wing, CC discretization is almost as accurate as the Residual Distribution Scheme (implemented in the VKI Thor code).

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