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MONOTONE DOMAIN DECOMPOSITION SCHEMES FOR A SINGULARLY PERTURBED SEMILINEAR ELLIPTIC REACTION-DIFFUSION EQUATION WITH ROBIN BOUNDARY CONDITIONS [1]

I.V. TSELISHCHEVA and G.I. SHISHKIN

Institute of Mathematics and Mechanics

Russian Academy of Sciences, 620219 Ekaterinburg, GSP-384, Russia

E-mail: `tsi@imm.uran.ru; shishkin@imm.uran.ru`

Abstract. We consider a semilinear singularly perturbed elliptic reaction-diffusion problem on a strip with Robin boundary conditions. We study a special (base) scheme comprising a standard difference operator on a piecewise-uniform mesh and an overlapping domain decomposition scheme. For these nonlinear schemes we construct monotone linearized schemes of the same ε-uniform accuracy. We apply the technique of upper and lower solutions to find *a posteriori* the number of iterations in the linearized scheme under which the accuracy of its solution is the same as for the base scheme. The number of required iterations is independent of ε. With respect to total computing costs, the method is close to a method for linear problems, since the number of iterations is only weakly depending on the number of mesh points. The decomposition schemes can be computed sequentially and in parallel.

Key words: singularly perturbed problem, elliptic reaction-diffusion equation, difference scheme, parameter-uniform convergence, domain decomposition method

1. Problem Formulation

On the vertical strip $\overline{D} = D \cup \Gamma$, $D = \{x : x_1 \in (0, d), x_2 \in R\}$, we consider the third boundary value problem for the semilinear singularly perturbed elliptic equation of reaction-diffusion type

$$L_{(1.1)}(u(x)) \equiv L^2_{(1.1)} u(x) - f(x, u(x)) = 0, \quad x \in D, \tag{1.1a}$$

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$$l\,u(x) \equiv \left\{\varepsilon\alpha(x)\frac{\partial}{\partial n} + \beta(x)\right\} u(x) = \psi(x), \quad x \in \Gamma. \tag{1.1b}$$

Here $L^2_{(1.1)} \equiv \varepsilon^2 \sum\limits_{s=1,2} a_s(x)\,\frac{\partial^2}{\partial x_s^2} - c(x)$, $\varepsilon \in (0,1]$ is the singular perturbation parameter, n denotes the outward normal to the boundary Γ. The functions $a_s(x)$, $c(x)$ and $f(x,u)$ are assumed to be sufficiently smooth on $\overline{D}$ and $\overline{D}\times R$, respectively, $\alpha(x)$, $\beta(x)$ and $\psi(x)$ are sufficiently smooth on Γ, and also [1]

$$\begin{gathered} 0 < a_0 \le a_s(x) \le a^0, \quad s = 1,2, \quad 0 < c_0 \le c(x) \le c^0, \quad x \in \overline{D}; \\ |f(x,u)| \le M, \quad 0 < c_1 \le c(x) + \frac{\partial}{\partial u} f(x,u) \le c^1, \quad (x,u) \in \overline{D}\times R; \\ 0 \le \alpha(x),\ \beta(x) \le M, \quad \alpha(x)+\beta(x) \ge m, \quad |\,\psi(x)\,| \le M, \quad x \in \Gamma. \end{gathered} \tag{1.2}$$

We have the Dirichlet problem if $\alpha(x) \equiv 0$ on Γ and the Neumann problem if $\beta(x) \equiv 0$ on Γ. As $\varepsilon \to 0$, a boundary layer arises in a neighborhood of Γ.

It is required to construct a base difference scheme and a scheme based on successive approximations and, for such schemes, to develop domain decomposition schemes. It is necessary that these schemes converge ε-uniformly, and the number of iterations required for convergence is independent of ε.

Further, we assume for simplicity that

$$\text{either} \quad \alpha(x) = 0 \quad \text{or} \quad \alpha(x) \ge m, \quad x \in \Gamma. \tag{1.3}$$

2. Base Finite Difference Scheme for Problem (1.1)

On the set $\overline{D}$ we introduce a rectangular grid

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{2.1}$$

where $\overline{\omega}_1$ and ω_2 are generally arbitrary nonuniform meshes on $[0,d]$ and on the axis x_2, respectively. Let $h_s^i = x_s^{i+1} - x_s^i$, $s = 1,2$, $x_1^i, x_1^{i+1} \in \overline{\omega}_1$, $x_2^i, x_2^{i+1} \in \omega_2$; let $h_s = \max_i h_s^i$, $h = \max_s h_s$. Assume $h \le MN^{-1}$, where $N = \min_s N_s$, N_1+1 and N_2+1 are the number of nodes in the mesh $\overline{\omega}_1$ and the minimal number of nodes in ω_2 on a unit interval of the axis x_2.

We approximate problem (1.1) by the difference scheme [3]

$$\begin{gathered} \Lambda_{(2.2)}(z(x)) \equiv \Lambda^2_{(2.2)} z(x) - f(x, z(x)) = 0, \quad x \in D_h, \\ \lambda^*_{(2.2)}\, z(x) = \psi^*(x;\, z(x)), \quad x \in \Gamma_h. \end{gathered} \tag{2.2}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\Lambda^2_{(2.2)} \equiv \varepsilon^2 \sum\limits_{s=1,2} a_s(x)\,\delta_{\overline{xs}\,\widehat{xs}} - c(x)$,

$$\lambda^*_{(2.2)} = \lambda_{(2.2)} + 2^{-1}\varepsilon^{-1}\alpha(x)\,h_1^*\,a_1^{-1}(x)\,[-\varepsilon^2 a_2(x)\delta_{\overline{x2}\,\widehat{x2}} + c(x)], \quad x \in \Gamma_h,$$

$$\lambda_{(2.2)} \equiv \varepsilon\alpha(x) \left\{ \begin{array}{l} -\delta_{x1}\,, \ \ x_1 = 0, \ \ x_2 \in R, \\ \ \ \delta_{\overline{x1}}\,, \ \ x_1 = d, \ \ x_2 \in R \end{array} \right\} + \beta(x),$$

$$\psi^*(x;\, z(x)) = \psi(x) - 2^{-1}\varepsilon^{-1}\alpha(x)\,h_1^*\,a_1^{-1}(x)\,f(x, z(x)), \quad x \in \Gamma_h,$$

$$h_1^* = h_1^1 \quad \text{for} \quad x_1 = 0, \qquad h_1^* = h_1^{N_1} \quad \text{for} \quad x_1 = d,$$

[1] Here and below M, M_i (or m) denote generic, sufficiently large (small) positive constants that are independent of ε and the discretization parameters.

$\delta_{\overline{xs}\,\widehat{xs}}z(x)=z_{\overline{xs}\,\widehat{xs}}(x)$ and $\delta_{x1}z(x)=z_{x1}(x)$, $\delta_{\overline{x1}}z(x)=z_{\overline{x1}}(x)$ are the second and first (forward and backward) difference derivatives on a nonuniform mesh [3].

The nonlinear scheme (2.2), (2.1) is ε-uniformly monotone [3]. On the uniform (in x_1 and x_2) grid $\overline{D}_h^u=\overline{\omega}_1\times\omega_2$, we have the error bound

$$|u(x)-z(x)|\le M\left[(\varepsilon+N_1^{-1})^{-2}N_1^{-2}+N_2^{-2}\right],\quad x\in\overline{D}_h^u.$$

Thus, scheme (2.2) on uniform meshes converges only for fixed values of ε.

Let us construct the base scheme that converges ε-uniformly [2, 4].

On the set $\overline{D}$, we place a special grid condensing in the layer regions s

$$\overline{D}_h=\overline{D}_h^\star=\overline{\omega}_1^\star\times\omega_2, \tag{2.3}$$

where ω_2 is a uniform mesh, $\overline{\omega}_1^\star=\overline{\omega}_1^\star(\sigma)$ is a piecewise uniform mesh. We divide $[0,d]$ into three parts $[0,\sigma]$, $[\sigma,d-\sigma]$ and $[d-\sigma,d]$. The mesh size on each subinterval is constant and equals $h_1^{(1)}=4\sigma/N_1$ on $[0,\sigma]$ and $[d-\sigma,d]$, and $h_1^{(2)}=2(d-2\sigma)/N_1$ on $[\sigma,d-\sigma]$. The parameter σ is defined by $\sigma=\sigma(\varepsilon,N_1)=\min[d/4,\ M_1\varepsilon\ln N_1]$, where $M_1\ge m^{-1}$, $0<m<m_0$, $m_0=((a^0)^{-1}c_1)^{1/2}$.

Theorem 1. *Let the data of problem* (1.1) *satisfy* (1.2) *and* (1.3)*, and also* a_1, $a_2,c\in C^{l+\alpha}(\overline{D})$, $f\in C^{l+\alpha}(\overline{D}\times R)$, $\alpha,\beta,\psi\in C^{l+\alpha}(\Gamma)$, $l=6$, $\alpha>0$. *Then the difference scheme* (2.2), (2.3) *converges* ε*-uniformly with the error bound*

$$|u(x)-z(x)|\le M\left[N_1^{-2}\ln^2 N_1+N_2^{-2}\right],\qquad x\in\overline{D}_h^\star. \tag{2.4}$$

3. Linearized Iterative Difference Scheme

To linearize scheme (2.2), we construct an iterative scheme in which the unknown function in the nonlinear terms is taken at the previous iteration:

$$\begin{aligned}&\Lambda_{(3.1)}(z(x,t))\equiv\Lambda^2_{(2.2)}z(x,t)-p\,\delta_{\overline{t}}\,z(x,t)-f(x,\check{z}(x,t))=0,\ \ (x,t)\in G_h,\\&\quad\check{\lambda}^*\,z(x,t)=\psi^*(x;\,\check{z}(x,t)),\ \ x\in S_h^L,\quad z(x,t)=\varphi(x),\ \ (x,t)\in S_{h0}.\end{aligned}\tag{3.1a}$$

Here

$$\overline{G}_h=G_h\cup S_h,\quad \overline{G}_h=\overline{D}_h\times\overline{\omega}_0,\quad G_h=D_h\times\omega_0, \tag{3.1b}$$

$\overline{\omega}_0$ is a uniform time-like mesh on the semiaxis $t\ge0$ with step-size $h_t=1$, the variable $t=0,1,\ldots\in\overline{\omega}_0$ specifies the iteration number; $S_h=S_h^L\cup S_{h0}$, $S_h^L=\Gamma_h\times\omega_0$ is the lateral boundary of $\overline{G}_h$; $\delta_{\overline{t}}z(x,t)=h_t^{-1}[z(x,t)-\check{z}(x,t)]$, $\check{z}(x,t)=z(x,t-h_t)$, $(x,t)\in G_h$; in $\psi^*(x)$ we also take $f(x,\check{z}(x,t))$;

$$\check{\lambda}^*\equiv\lambda_{(2.2)}+\,2^{-1}\varepsilon^{-1}\alpha(x)\,h_1^*\,a_1^{-1}(x)\left[-\varepsilon^2a_2(x)\delta_{\overline{x2}\,\widehat{x2}}+c(x)+p\,\delta_{\overline{t}}\right],\quad x\in\Gamma_h;$$

the coefficient p satisfies the condition $p-\dfrac{\partial}{\partial u}f(x,u)\ge p_0>0$, $(x,u)\in\overline{D}\times R$; the initial data $\varphi(x)$, $x\in\overline{D}_h$, is a sufficiently smooth bounded function satisfying $\lambda\,\varphi(x)=\psi(x)$, $x\in\Gamma_h$. The function $z(x,t)$, $(x,t)\in\overline{G}_h$, where $\overline{G}_h$ is generated by $\overline{D}_{h(2.1)}$, is called the solution of scheme (3.1), (2.1).

Scheme (3.1), (2.1) linear at each iteration is monotone. Its solution $z(x,t)$ converges ε-uniformly to the solution of the base scheme (2.2), (2.1) at a rate of geometric progression. On the mesh (2.3), we obtain the ε-uniform bound

$$|u(x)-z(x,t)|\le M\left[N_1^{-2}\ln^2 N_1+N_2^{-2}+q_1^t\right],\quad (x,t)\in\overline{G}_h^*. \tag{3.2}$$

Here $\overline{G}_h^* = \overline{G}_h(\overline{D}_{h(2.3)}^*)$, $q_1 \le p^0(c_1 + p^0)^{-1}$, $p^0 = \max\big(p - \frac{\partial}{\partial u} f(x,u)\big)$, $c_1 = \min\big(c(x) + \frac{\partial}{\partial u} f(x,u)\big)$, $(x,u) \in \overline{D} \times R$.

4. Iterative Difference Scheme of the Schwarz Method

4.1. For scheme (2.2), we describe an overlapping domain decomposition method [1, 6]. Let open subdomains D^k, $k = 1, \dots, K$ cover the domain D: $D = \bigcup_{k=1}^K D^k$. Denote the minimal overlap of the sets D^k and $D^{[k]} = \bigcup_{i=1,\ i \ne k}^K D^i$ by $\varDelta^k$, and the smallest value of $\varDelta^k$ by $\varDelta$, $k = 1, \dots, K$, i.e.

$$\min_{k,\ x^1,\ x^2} \rho(x^1, x^2) = \varDelta, \quad x^1 \in \overline{D}^k, \quad x^2 \in \overline{D}^{[k]}, \quad x^1,\ x^2 \notin \{D^k \cap D^{[k]}\},$$

where $\rho(x^1, x^2)$ is the distance between x^1 and x^2. Generally, $\varDelta = \varDelta(\varepsilon)$.

It is convenient to introduce the uniform "time" mesh $\omega_0 = \{t^n :\ t^n = nh_t,\ n = 1, 2, \dots\}$, $\overline{\omega}_0 = \omega_0 \cup \{t = 0\}$ with step-size $h_t = 1$, by associating its nodes to the iteration number, and thus the semi-discrete set $\overline{G} = \overline{D} \times \overline{\omega}_0$, $G = D \times \omega_0$, with the boundary $S = S^L \cup S_0$, $S^L = \varGamma \times \omega_0$ being the lateral boundary.

Let each set D^k be partitioned into P disjoint (possibly empty) sets:

$$D^k = \bigcup_{p=1}^P D_p^k, \quad k = 1, \dots, K, \quad \overline{D}_i^k \bigcap \overline{D}_j^k = \emptyset, \quad i \ne j.$$

We set $G_p^k = D_p^k \times \overline{\omega}_0$, $p = 1, \dots, P$, $k = 1, \dots, K$. On the sets $\overline{G}$, $\overline{G}_p^k$ we construct the grids

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{G}_{ph}^k = \overline{D}_{ph}^k \times \overline{\omega}_0, \quad \overline{D}_{ph}^k = \overline{D}_p^k \bigcap \overline{D}_h \tag{4.1a}$$

where $\overline{D}_h$ is grid (2.1) or (2.3); we suppose that the faces of $\overline{D}_p^k$ pass through the nodes of the grid $\overline{D}_h$. For $t = 0$ we define the function $z(x,t)$ by $z(x,0) = \varphi(x)$, $x \in \overline{D}_h$, where $\varphi(x)$ is a sufficiently arbitrary bounded function satisfying $\lambda\,\varphi(x) = \psi(x)$, $x \in \varGamma_h$.

We now determine the sequence of discrete functions $z(x,t)$, $(x,t) \in \overline{G}_h$, $t = 1, 2, \dots$. Before we find the sequence of auxiliary functions $z^{\frac{k}{K}}(x,t)$, $x \in \overline{D}_h$, $k = 1, \dots, K$, $t = 1, 2, \dots$, by solving the boundary value problems

$$\left.\begin{aligned} \varLambda_{(2.2)}\big(z_p^{\frac{k}{K}}(x,t)\big) &= 0, && x \in D_{ph}^k, \\ z_p^{\frac{k}{K}}(x,t) &= z^{\frac{k-1}{K}}(x,t), && x \in \varGamma_{ph}^k \setminus \varGamma_h, \\ \lambda_{(2.2)}^*\, z_p^{\frac{k}{K}}(x,t) &= \psi^*(x;\, z_p^{\frac{k}{K}}(x,t)), && x \in \varGamma_{ph}^k \cap \varGamma_h, \end{aligned}\right\}, \quad p = 1, \dots, P; \tag{4.1b}$$

$$z^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} z_p^{\frac{k}{K}}(x,t), & x \in \overline{D}_{ph}^k, \quad p = 1, \dots, P, \\ z^{\frac{k-1}{K}}(x,t), & x \in \overline{D}_h \setminus \overline{D}^k \end{array}\right\}, \quad x \in \overline{D}_h, \quad k = 1, \dots, K;$$

$$z^{\frac{k-1}{K}}(x,t) = z(x,t-1), \quad x \in \overline{D}_h \quad \text{for} \quad k = 1;$$

$$z(x,0) = \varphi(x), \quad x \in \overline{D}_h;$$

$$z(x,t) = z^{\frac{K}{K}}(x,t); \qquad t = 1, 2, \dots\ ; \tag{4.1c}$$

here $\varGamma_{ph}^k = \overline{D}_{ph}^k \setminus D_{ph}^k$. It is required to find the sequence of functions $z(x,t)$,

$x \in \overline{D}_h,\ t = 1, 2, ...,$ which are solutions of the iterative scheme (4.1), (2.1) (or (4.1), (2.3)) using $P > 1$ solvers. The intermediate problems (4.1b) on the subdomains $\overline{D}^k_{ph}$, $p = 1, ..., P$ can be solved in parallel, independently of each other, on P processors [1]. For $P = 1$ the problems on $\overline{D}^k_h = \overline{D}^k \cap \overline{D}_h$ are solved sequentially. Scheme (4.1), (2.1) is nonlinear.

For

$$\Delta = \Delta(\varepsilon) > 0, \quad \varepsilon \in (0, 1], \qquad \inf_{\varepsilon \in (0,1]} \left[\varepsilon^{-1} \Delta(\varepsilon) \right] \geq m_1 > 0 \tag{4.2}$$

$z(x, t)$ as $t \to \infty$ converges to the solution $z(x)$ of scheme (2.2), (2.1) ε-uniformly

$$|z(x) - z(x, t)| \leq M\, q^t, \quad (x, t) \in \overline{G}_h, \quad \text{where} \quad q \leq 1 - m, \tag{4.3}$$

$q = q(m_1)$, and $q(m_1)$ grows as $m_1 \to 0$; in general, $q > q_{1(3.2)}$. Condition (4.2) is necessary and sufficient for the ε-uniform convergence (as $t \to \infty$) of the solutions of the iterative scheme (4.1) to the solution of the base scheme (2.2). Taking into account estimates (4.3) and (2.4), for the solution of scheme (4.1), (2.3) we find the estimate similar to (3.2)

$$|u(x) - z(x, t)| \leq M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} + q_2^t \right], \ (x, t) \in \overline{G}^*_h, \ q_2 \leq 1 - m, \tag{4.4}$$

where, generally speaking, $q_2 > q_{(4.3)}$. For a linear problem, $q_2 = q_{(4.3)}$.

Theorem 2. *Let the hypothesis of Theorem* 1 *and condition* (4.2) *hold. Then the Schwarz scheme* (4.1)*,* (2.3) *as* $N_1, N_2, t \to \infty$ *converges* ε*-uniformly at the rate* $O\big(N_1^{-2} \ln^2 N_1 + N_2^{-2} + q_2^t\big)$*,* $q_2 \leq 1 - m$*, with the error bound* (4.4)*.*

4.2. Based on the linearized scheme (3.1), (2.3), it is possible to construct the monotone linearized scheme of the Schwarz method that converges at the same rate $O\big(N_1^{-2} \ln^2 N_1 + N_2^{-2} + q^t\big)$, $q \leq 1 - m$. When solving the subproblem at the intermediate (inner) iteration in this scheme, the unknown function in the nonlinear terms is taken at the previous iteration.

Let $z(x, t)$, $(x, t) \in \overline{G}_h$ be a solution of the discrete Schwarz method, $z^{(j)}(x, t)$, $(x, t) \in \overline{G}_h$, $j = 1, 2$, be solutions of some difference scheme, and let the following inequality be satisfied for $t = 0$:

$$z^{(1)}(x, 0) \leq z(x, 0) \leq z^{(2)}(x, 0), \quad x \in \overline{D}_h. \tag{4.5}$$

If the inequality $z^{(1)}(x, t) \leq z(x, t) \leq z^{(2)}(x, t)$, $(x, t) \in \overline{G}_h$, is true for $t > 0$, and also

$$\max_x \left| z^{(i)}(x, t) - z(x, t) \right| \to 0, \quad x \in \overline{D}_h \quad \text{for} \quad t \to \infty, \quad i = 1, 2,$$

we call the functions $z^{(1)}(x, t)$ and $z^{(2)}(x, t)$ the lower and upper solutions of the discrete Schwarz method.

Because the Schwarz scheme is monotone, its solutions $z^{(1)}(x, t)$ and $z^{(2)}(x, t)$, $(x, t) \in \overline{G}_h$, satisfying condition (4.5) for $t = 0$ are the lower and upper solutions. We use upper and lower solutions to evaluate *a posteriori* the number of iterations for which the accuracy of the linearized scheme is the same (up to a factor) as that for the base scheme (2.2), (2.3) (see also [5]).

The error in the solution of the linearized Schwarz scheme on the mesh (2.3) can be represented in the form

$$z^{(j)}(x,t) - u(x) = \big(z(x) - u(x)\big) + \big(z^{(j)}(x,t) - z(x)\big), \quad (x,t) \in \overline{G}_h, \quad j = 1, 2,$$

where $z^{(j)}(x,t)$ is the solution of the linearized Schwarz scheme satisfying the condition $z^{(1)}(x,0) \le z(x) \le z^{(2)}(x,0), \quad x \in \overline{D}_h$.

Let T be the number of iterations (in t) in the linearized Schwarz scheme under which the error in the solution of the base scheme (2.2), (2.3) and the deviation of the solution of the linearized scheme from the solution of the base scheme are commensurable. We call the function $z^{(j)}(x,T),\ x \in \overline{D}_h$, the solution (upper for $j = 2$ and lower for $j = 1$) of the linearized Schwarz scheme, *consistent* with respect to the accuracy (of the base scheme) and with respect to the number of iterations (of the linearized Schwarz scheme).

For the upper and lower solutions of the linearized Schwarz scheme on the mesh (2.3), we find the least value of T for which such a condition holds:

$$\max_{\overline{D}_h} \left[z^{(2)}(x,T) - z^{(1)}(x,T)\right] \le M_1 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2}\right], \quad x \in \overline{D}_h.$$

For the consistent solution of this scheme we obtain the estimate

$$|u(x) - z^{(j)}(x,T)| \le M_2 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2}\right], \quad x \in \overline{D}_h, \ \ j = 1, 2,$$

with T satisfying the inequality

$$T \le M_3 \big(\ln q^{-1}\big)^{-1} \ln \big(\min\,[N_1, N_2]\big),$$

where $q \le 1 - m$, the constants M_1, M_2, M_3 are independent of q.

Thus, the number of required iterations is independent of ε. With respect to total computational costs, the iterative method is close to a solution method for linear problems, since the number of iterations is only weakly depending on the number of mesh points used.

Remark 1. For $p = 0$ the iterates in the linearized scheme as $t \to \infty$, in general, diverge, for example, under the condition $(\partial/\partial u) f(x,u) > c(x),\ (x,u) \in \overline{D} \times R$.

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