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POINTWISE BOUNDS ON DERIVATIVES OF SOLUTIONS OF ELLIPTIC CONVECTION-DIFFUSION PROBLEMS [1]

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Abstract. An accessible overview will be presented of some rather technical results from two recent papers of the authors [3, 4] that derive pointwise bounds on derivatives of solutions to singularly perturbed elliptic boundary value problems posed in two dimensions. The dependence of these bounds on the regularity and compatibility of the data is our main interest. In particular we focus on the effects of data incompatibility at the corners when the domain is the unit square and of a discontinuity of some (possibly zero-order) derivative of the solution at a point on the inflow boundary. The bounds show the effects of a parabolic boundary or interior layer in the solution. The results should be useful in devising mesh refinement strategies for a numerical solution of these problems.

Key words: singular perturbation, convection-diffusion, corner singularity, boundary layer, interior layer

1. Introduction

Singularly perturbed convection-diffusion problems arise in many applications. Their solutions typically exhibit boundary and interior layers and the asymptotic nature of these layers has been widely studied; see the references in [7]. The numerical solution of convection-diffusion problems is also the subject of intensive investigation — an introduction is given in [10] while [7] gives an extensive overview.

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To derive sharp error bounds in the rigorous analysis of a numerical method for these problems one needs precise information on how derivatives of the solution of the convection-diffusion problem depend on the singular perturbation parameter, but in general this information is not furnished by asymptotic expansions. Such a priori bounds on derivatives are well-known in singularly perturbed 2-point boundary value problems [7] but their derivation for problems posed in two dimensions is more difficult.

For elliptic convection-diffusion problems on bounded domains in $\mathbb{R}^2$, estimates of global Sobolev norms of u and some pointwise bounds are given in [1]. Problems posed on the unit square are considered in the following papers:

- in [5] pointwise derivative bounds are proved for a problem whose solution u has exponential boundary layers along two sides of the square;
- [8] is mainly concerned with asymptotic expansions for a problem on the unit square, but some derivative bounds are proved that exhibit a parabolic boundary layer in u;
- the derivative bounds in [6] involve both exponential and parabolic boundary layers but some of the arguments are incomplete.

Section 2 below will present a simplified version of the results from [4] where we consider a unit square convection-diffusion problem whose solution has boundary and interior layers and derive derivative bounds via a detailed analysis related to that of [6]; furthermore, we also consider the effects of incompatibilities in the problem data at the corners of the square.

None of the above papers gives any information about the pointwise behavior of the solution u near interior layers. This question is tackled in [9, Chapter IV], but many details are omitted and it is difficult to establish the precise assumptions made. In the present paper Section 3 discusses the pointwise derivative bounds from [3], where we study a convection-diffusion problem on a half-plane whose boundary data (or some derivative of it) has a discontinuity at one point, resulting in an interior layer along the subcharacteristic that emanates from that point.

Remark. The singular perturbation parameter will be denoted by ε; it lies in the interval $(0, 1]$. We use C to denote a generic positive constant that in particular is independent of ε and of the location of the point $(x, y) \in Q$.

2. Unit Square Problem

Let Q be the open unit square in $\mathbb{R}^2$, with boundary ∂Q. Let u be the solution of the elliptic convection-diffusion problem

$$-\varepsilon \Delta u + p u_x + q u = f \text{ in } Q, \tag{2.1a}$$

$$u = g \text{ on } \partial Q, \tag{2.1b}$$

where p and q are positive constants and $g = (g_s, g_e, g_n, g_w)$ where these are respectively the restriction of g to the south, east, north and west sides of ∂Q. Assume that $f \in C^{2\ell,\alpha}(\bar{Q})$ and $g_w, g_e, g_s, g_n \in C^{2\ell,\alpha}[0,1]$ for some non-negative integer ℓ and $\alpha \in (0,1)$, where $C^{2\ell,\alpha}$ is the usual Hölder space.

In [4] we allow the possibility that g is discontinuous at the corners of ∂Q or that some higher-order compatibility condition may fail; see [2] for a discussion of compatibility conditions for (2.1). Pointwise bounds on the derivatives of u are then proved under the hypothesis of an arbitrary but fixed degree of compatibility at each corner.

HYPOTHESIS 1. For brevity here we shall assume that the function g is discontinuous at each corner, i.e. that one has no compatibility in the data.

In (2.1) the flow is parallel to the x-axis, from left to right, so from asymptotic considerations one typically expects the following behaviour in the solution u: an exponential boundary layer at the outflow boundary $x = 1$, and parabolic boundary layers along the characteristic boundaries $y = 0$ and $y = 1$. (For an introductory discussion of these phenomena see [10]; more details are given in [7].) Nevertheless pointwise bounds confirming these expectations were first proved only recently, in [4]. And there is a further complication: in non-singularly perturbed problems the incompatibility of the data is known to yield a certain singularity at each corner, but how will the presence of the small parameter ε affect this singularity?

Let $(x, y) \in Q$. Set r_{ij} = distance from (x, y) to (i, j) for $i, j = 0, 1$. Set $r = \min\{r_{00}, r_{01}, r_{10}, r_{11}\}$, so r is the distance from (x, y) to the nearest corner of ∂Q. Let m and n be non-negative integers with $m + n > 0$, $2m + n \le 2\ell$ and $m + n \le 2\ell - 2$. Let $\bar{n}$ be the smallest even integer that satisfies $\bar{n} \ge n$.

The following derivative bounds are proved in [4]. They depend in nature on whether or not one is near a corner of ∂Q. In the theorem the *inflow corners* are (0,0) and (0,1) while the *outflow corners* are (1,0) and (1,1).

Theorem 1.

(i) Suppose that $r \le \varepsilon$ so (x, y) is close to a corner. Then

$$|D_x^m D_y^n u(x,y)| \le C r^{-m-n} \quad \textit{near inflow corners,}$$
$$|D_x^m D_y^n u(x,y)| \le C \varepsilon^{-2m-\bar{n}} r^{-m-n} \quad \textit{near outflow corners.}$$

(ii) Suppose that $r > \varepsilon$ so (x, y) is not close to a corner. Choose a constant β such that $0 < \beta < \min\{\sqrt{q}, p\}$. Then

$$\begin{aligned} |D_x^m D_y^n u(x,y)| \le C \Big\{ & 1 + \varepsilon^{-m} e^{-p(1-x)/\varepsilon} \\ & + \varepsilon^{-n/2} \left[r_{00}^{-m-n/2} e^{-\beta y/\sqrt{\varepsilon}} + r_{01}^{-m-n/2} e^{-\beta(1-y)/\sqrt{\varepsilon}} \right] \\ & + \varepsilon^{-2m-\bar{n}} \varepsilon^{-n/2} \left[1 + r_{10}^{-m-n/2} \right] e^{-p(1-x)/\varepsilon} e^{-\beta y/\sqrt{\varepsilon}} \\ & + \varepsilon^{-2m-\bar{n}} \varepsilon^{-n/2} \left[1 + r_{11}^{-m-n/2} \right] e^{-p(1-x)/\varepsilon} e^{-\beta(1-y)/\sqrt{\varepsilon}} \Big\}. \end{aligned}$$

The bounds of part (i) show that near an inflow corner the small parameter ε has no effect: one simply obtains the classical singularity in the solution u.

But near an outflow corner on the other hand, the smallness of ε amplifies the classical singularity. A heuristic explanation for this dichotomy is that at the inflow corners the flow points away from the corner and consequently induces no layer effect, but the opposite is true at the outflow corners.

The bound of part (ii), which holds sway on most of Q, is also interesting. The terms $1 + \varepsilon^{-m} e^{-p(1-x)/\varepsilon}$ are to be expected: they represent the effect of the reduced solution of (2.1) on Q and the one-dimensional-like exponential layer at the outflow boundary $x = 1$. The factors depending on n in the term $\varepsilon^{-n/2} r_{00}^{-m-n/2} e^{-\beta y/\sqrt{\varepsilon}}$ describe the parabolic boundary layer along the side $y = 0$, while the factor r_{00}^{-m}, which is influential only in a neighbourhood of the corner (0,0), means that this bound blends into the bound of part (i) as r nears (0,0). The term $\varepsilon^{-n/2} r_{01}^{-m-n/2} e^{-\beta(1-y)/\sqrt{\varepsilon}}$ is similarly associated with the parabolic boundary layer along the side $y = 1$. The remaining terms in the estimate of (ii) are corner layers at (1,0) and (1,1) that come from the interaction between the exponential outflow layer along $x = 1$ and the parabolic layers just mentioned; observe the same multiplicative factor $\varepsilon^{-2m-\bar{n}}$ as in the outflow corners of part (i).

The bounds of parts (i) and (ii) are equivalent when $r = \varepsilon$ in both.

Proof. (Outline) The solution u is decomposed as a sum of terms that bear a superficial resemblance to a standard asymptotic expansion (e.g. one term corresponds to the reduced solution, another to the exponential outflow layer, etc.), but each term is defined as the solution to an elliptic boundary-value problem in a half-plane or quarter-plane. The solutions of the quarter-plane problems are in turn decomposed as sums of solutions to half-plane problems. The derivatives of the solutions to most of these elliptic problems are bounded using maximum principles and barrier functions combined with induction. The exception is the elliptic problem associated with each corner singularity, where we use an explicit representation of the solution (cf. (3.2) below) — this is available since the differential operator has constant coefficients. By a lengthy calculation one can bound the pure y-derivatives in this problem, then an inductive argument yields bounds on the remaining derivatives. ■

3. Half-Plane Problem

The second problem that we consider is simpler insofar as the singular perturbation parameter gives rise to only one phenomenon — a parabolic interior layer — in the solution u, but such layers are tricky to analyse.

Let $\mathbb{R}^2_+$ denote the right-hand half-plane. Consider the problem

$$Lu := -\varepsilon \Delta u + p_1 u_x + p_2 u_y + qu = f \quad \text{for } (x, y) \in \mathbb{R}^2_+, \tag{3.1a}$$

$$u(0, y) = h(y) \quad \text{for } y \in \mathbb{R}. \tag{3.1b}$$

Here p_1 and q are positive constants, while p_2 is any constant (it may even be zero). Assume that $h \in C^\infty[0, \infty)$ and h has a smooth extension from $(-\infty, 0)$ to $(-\infty, 0]$, but $h(y)$ or one of its derivatives may have a jump discontinuity at $y = 0$. This discontinuity will induce a characteristic interior layer in u along the subcharacteristic passing through (0,0).

An integer-valued parameter $\nu \geq -1$ is used to indicate the degree of discontinuity that $h(y)$ has at $y = 0$: $\nu = -1$ means that $h(+0) \neq h(-0)$, while $\nu \geq 0$ means that

$$D^k h(+0) = D^k h(-0) \quad \text{for } k = 0, \cdots, \nu.$$

Let $\alpha = [p_1, p_2]$ be the subcharacteristic direction. Set $\beta = [-p_2, p_1]$ so $\beta \perp \alpha$ is the crosswind direction. It is more convenient in the analysis of this problem to examine directional derivatives D_α and D_β with respect to α and β than the derivatives D_x and D_y.

Let m and n be non-negative integers. Let $f \in H^{m+n+2}(\mathbb{R}^2_+)$, where $H^k(\cdot)$ is the usual Sobolev norm. The following two theorems are proved in [3].

Theorem 2. *(bounds on low-order derivatives) If $h \in H^{m+n+1}(-\infty, 0)$, $h \in H^{m+n+1}(0, \infty)$ and $n \leq \nu$, then $|D^m_\alpha D^n_\beta u(x, y)| \leq C$.*

Proof. (Outline) Let F be an extension of f to $\mathbb{R}^2$. Consider $LU = F$ in $\mathbb{R}^2$. From Sobolev and energy inequalities one has

$$|D^m_\alpha D^n_\beta U(x, y)| \leq \|U\|_{H^{m+n+2}(\mathbb{R}^2)} \leq C\|F\|_{H^{m+n+2}(\mathbb{R}^2)} \leq C\|f\|_{H^{m+n+2}(\mathbb{R}^2_+)}.$$

This reduces (3.1) to the case $f \equiv 0$. Now invoke a Fourier transform in y, solve the resulting initial-value problem, then use Parseval's formula and Sobolev's inequality in one dimension to bound the desired derivatives. ■

Theorem 2 confirms the intuitive expectations that derivatives D^m_α along the subcharacteristic should not depend on ε, and when the order of the crosswind derivative D^n_β is at most the degree of discontinuity, no layer is visible.

Theorem 3. *(bounds on higher-order derivatives) Let $f \in H^{m+n+2}(\mathbb{R}^2_+)$. If $h \in H^{2m+n+1}(-\infty, 0)$, $h \in H^{2m+n+1}(0, \infty)$ and $\nu < n$, then*

$$|D^m_\alpha D^n_\beta u(x, y)| \leq \begin{cases} C(1 + r^{-m-n+\nu+1}) & \text{for } r \leq 2\varepsilon, \\ C\Big[1 + \varepsilon^{(-n+\nu+1)/2} r^{-m+(-n+\nu+1)/2} e^{-cd^2/\varepsilon} & \\ \qquad\qquad + r^{-m-n+\nu+1} e^{-cr/\varepsilon}\Big] & \text{for } r \geq 2\varepsilon, \end{cases}$$

where d denotes the perpendicular distance from (x, y) to the subcharacteristic $p_1 y = p_2 x$ passing through (0,0).

Proof. (Outline) For $x > 0$ one has

$$u(x, y) = \frac{x}{2\pi\varepsilon} \int_0^\infty h(t)\zeta_1(t) \frac{1}{r_1(t)} K_1\left(\frac{\kappa r_1(t)}{2\varepsilon}\right) dt, \tag{3.2}$$

where $r_1(t) = \sqrt{x^2 + (y - t)^2}$, the function $K_1(\cdot)$ is a modified Bessel function of the second kind, $\kappa = \sqrt{p_1^2 + p_2^2 + 4\varepsilon q}$ and $\zeta_1(t) = e^{(p_1 x + p_2(y-t))/(2\varepsilon)}$. For the pure crosswind derivatives $D^n_\beta u(x,y)$, differentiate this formula. Then bounds on the other derivatives can be deduced from (3.1a) by an inductive argument. ∎

As in Theorem 1 we see that the classical singularity of the solution u at (0,0) is unaffected by the singularly perturbed nature of the problem. As one moves away from (0,0), this singularity dies off rapidly because of the factor $e^{-cr/\varepsilon}$ (so the smallness of ε actually helps to smooth the solution) but simultaneously a parabolic interior layer appears, as indicated by the factor $\varepsilon^{(-n+\nu+1)/2}e^{-cd^2/\varepsilon}$.

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