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THE DISCRETE RICHARDSON METHOD FOR SEMILINEAR PARABOLIC SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATIONS [1]

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Abstract. The Dirichlet problem for a semilinear singularly perturbed parabolic convection-diffusion equation is considered on the interval. For such a problem the basic finite difference (nonlinear) scheme based on classical approximations on *piecewise uniform* meshes condensing in the layer, converges ε-uniformly at a rate which does not exceed 1. Using the Richardson technique, we construct a scheme convergent ε-uniformly at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-q})$, $q \geq 2$, where N and N_0 define the number of nodes in the spatial and time meshes, respectively.

Key words: singular perturbation, boundary layer, parabolic, quasilinear equation, discrete approximation, linearization, high-order accuracy, Richardson technique, ε-uniform convergence

1. Problem Formulation. Aim of the Research

On the set $\overline{G}$

$$\overline{G} = G \bigcup S, \quad G = D \times (0, T], \tag{1.1}$$

where $D = (0, d)$, we consider the Dirichlet problem for the quasilinear singularly perturbed parabolic convection-diffusion equation

$$L\big(u(x,t)\big) \equiv L^2\, u(x,t) - f\big(x, t,\, u(x,t)\big) = 0, \quad (x,t) \in G, \tag{1.2}$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S.$$

Here

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$$L^2 = \varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}, \quad (x,t) \in G,$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t,\,u)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$, $\overline{G} \times R$ and S respectively, moreover [2]

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad |c(x,t)| \le c^0, \tag{1.3}$$
$$p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G};$$

$$|f(x,t,\,u)| \le M, \quad c_1 \le c(x,t) + \frac{\partial}{\partial u} f(x,t,\,u) \le c^1, \quad (x,t,\,u) \in \overline{G} \times R;$$
$$|\varphi(x,t)| \le M, \quad x \in S; \qquad a_0,\, b_0,\, c_1,\, p_0 > 0;$$

the perturbation parameter ε takes arbitrary values in $(0,1]$. Assume that the data of problem (1.1), (1.2) on the set of corner points $S^* = S_0 \bigcap \overline{S}^L$ satisfy the compatibility conditions which ensure the required smoothness of the solution on $\overline{G}$ (see, e.g., [2]). Here $S = S_0 \bigcup S^L$, S_0 and S^L are the lower and lateral parts of the boundary; $S_0 = \overline{S}_0$. For small values of ε, a regular boundary layer appears in a neighbourhood of the set $S_1^L = \{(x,t):\ x = 0,\ 0 < t \le T\}$. Here S_1^L and S_2^L are the left and right parts of the lateral boundary; $S^L = S_1^L \bigcup S_2^L$.

Our aim is for the boundary value problem (1.1), (1.2), using the Richardson technique, to construct a finite difference scheme convergent ε-uniformly with accuracy higher than 1.

2. Basic Finite Difference Scheme

First we present a ε-uniformly convergent finite difference scheme constructed on the base of classical approximation of problem (1.1), (1.2).

On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{2.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are arbitrary, in general, nonuniform meshes on the segments $[0,d]$ and $[0,T]$ respectively. Let $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$, and $h_t^k = t^{k+1} - t^k$, t^k, $t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. Assume that the condition $h \le M\,N^{-1}$, $h_t \le M\,N_0^{-1}$ be satisfied, where $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively.

We approximate problem (1.2), (1.1) by the finite difference scheme [4]

$$\Lambda\,(z(x,t)) \equiv \Lambda^2\, z(x,t) - f(x,t,\, z(x,t)) = 0, \quad (x,t) \in G_h, \tag{2.2}$$
$$z(x,t) \ = \varphi(x,t), \qquad (x,t) \in S_h.$$

Here $G_h = G \bigcap \overline{G}_h$, $S_h = S \bigcap \overline{G}$,

[2] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters.

$$\Lambda^2 \equiv \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}, \quad (x,t) \in G_h,$$

$\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the central difference derivative on the nonuniform mesh

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)], \quad (x,t) = (x^i,t) \in G_h,$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first order (forward and backward) finite differences.

Now we construct the base scheme convergent ε-uniformly (see, e.g., [3, 6]). On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{2.3a}$$

where $\overline{\omega}_0$ is a uniform mesh, $\overline{\omega}^*$ is a special *piecewise* uniform mesh which is constructed as follows. The segment $[0,d]$ is divided in two parts $[0,\sigma]$, $[\sigma,d]$, step-sizes in these parts are constant and equal to $h^{(1)} = 2\,\sigma\, N^{-1}$ and $h^{(2)} = 2(d-\sigma)N^{-1}$ respectively. The parameter σ is defined by the relation

$$\sigma = \sigma(\varepsilon,\, N,\, l) = \min\left[2^{-1}\, d,\, l\, m^{-1}\, \varepsilon\, \ln N\right], \tag{2.3b}$$

where m is arbitrary number in $(0,m_0)$, $m_0 = \min_{\overline{G}}[a^{-1}(x,t)\, b(x,t)]$. Here

$$l = 1. \tag{2.3c}$$

For the other meshes this parameter will be chosen separately. The mesh $\overline{\omega}^*$ and mesh $\overline{G}_h = \overline{G}_h(l=1)$ are constructed.

For solutions of the difference scheme (2.2), (2.3) we obtain the following ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M\, \left[N^{-1}\, \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{2.4}$$

Thus, the order of ε-uniform convergence does not exceed 1.

3. The Richardson Scheme

Now we give the Richardson method used to improve accuracy for solutions of the proposed special basic scheme. This method was applied for improvement of accuracy to linear singularly perturbed problems (see, e.g., [1, 5, 7] and also the bibliography therein).

On the set $\overline{G}$ we construct meshes

$$\overline{G}_h^i = \overline{\omega}^{*i} \times \overline{\omega}_0^i, \quad i = 1,2, \tag{3.1a}$$

uniform in t and *piecewise uniform* in x. Here $\overline{G}_h^2$ is $\overline{G}_{h(2.3a)}$, where

$$\sigma = \sigma_{(2.3\mathrm{b})}(\varepsilon,\, N,\, l) \;\text{ for }\; l = 2; \tag{3.1b}$$

$\overline{G}_h^1$ is a coarsened mesh. For the parameters σ^i, which define piecewise uniform meshes $\overline{\omega}^{*i} = \overline{\omega}^{*i}(\sigma^i)$, we impose the condition $\sigma^1 = \sigma^2$, where $\sigma^2 = \sigma_{(3.1\mathrm{b})}$, that is, segments on which the meshes $\overline{\omega}^{*1}$ and $\overline{\omega}^{*2}$ have a constant step-size, are the same. Step-sizes in the mesh $\overline{\omega}^{*1}$ on the segments $[0,\sigma]$, $[\sigma,d]$ are k times larger than step-sizes in the mesh $\overline{\omega}^{*2}$, and the step-size in the mesh $\overline{\omega}_0^1$ (on the segment $[0,T]$) are k times larger than the step-size in the mesh $\overline{\omega}_0^2$ ($k^{-1}N+1$ and $k^{-1}N_0+1$ are the number of nodes in the meshes $\overline{\omega}^{*1}$ and $\overline{\omega}_0^1$ respectively). Let

$$\overline{G}_h^0 = \overline{G}_h^1 \bigcap \overline{G}_h^2. \tag{3.1c}$$

Let $z^i(x,t)$, $(x,t) \in \overline{G}_h^i$, $i = 1, 2$ be solutions of the difference schemes[3]

$$\Lambda_{(2.2)}\left(z^i(x,t)\right) = 0, \qquad (x,t) \in G_h^i, \tag{3.2a}$$

$$z^i(x,t) = \varphi(x,t), \quad (x,t) \in S_h^i, \quad i = 1,2. \tag{3.2b}$$

Assume

$$z^0(x,t) = \gamma\, z^1(x,t) + (1-\gamma)\, z^2(x,t), \quad (x,t) \in \overline{G}_h^0, \tag{3.2c}$$

where $\gamma = \gamma(k) = -(k-1)^{-1}$. We call the function $z^0_{(3.2)}(x,t)$, $(x,t) \in \overline{G}_h^0$ the solution of the difference scheme (3.2), (3.1), i.e. the scheme based on the Richardson method on two embedded meshes.

Taking into account *a-priori* estimates we obtain

$$|u(x,t) - z^0(x,t)| \le M\, \left[N^{-2}\ln^2 N + N_0^{-2}\right], \quad (x,t) \in \overline{G}_h^0. \tag{3.3}$$

Thus, the order of ε-uniform convergence is the second in t and the second up to a logarithmic factor in x.

Theorem 1. *Let for the data of the boundary value problem* (1.2), (1.1) *the condition* $a, b, c, p \in C^{6+\alpha}(\overline{G})$, $f \in C^{6+\alpha}(\overline{G}\times R)$, $\varphi \in C^{6+\alpha}(S)$, $\alpha > 0$, *and also condition* (1.3) *and the condition*

$$\varphi(x,t) = 0, \quad (x,t) \in S_0, \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad (x,t) \in S^*,$$

$$\frac{\partial^{k+k_0+k_u}}{\partial x^k\, \partial t^{k_0}\, \partial u^{k_u}} f(x,t,u) = 0, \quad (x,t) \in S^*, \quad u = 0, \quad k,\, k_0,\, k_u \le 6,$$

be fulfilled, and let for $u(x,t)$, *that is the solution of the problem, and for its the regular and singular components* $U(x,t)$ *and* $V(x,t)$ *the inclusion* $u, U, V \in C^6(\overline{G})$ *be satisfied. Then the solution of the difference scheme* (3.2), (3.1) *converges to the solution of the boundary value problem* ε*-uniformly with the rate* $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-2})$. *For the discrete solution the estimate* (3.3) *is valid.*

[3] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in formula ($j.k$).

4. Linearized Basic Scheme

On mesh (2.1) we consider a difference scheme in which the nonlinear term of the differential equation is computed using the sought function from the previous time level. To the boundary value problem (1.1), (1.2) corresponds the linearized difference scheme (see [4])

$$\Lambda_{(4.1)}\,(z(x,\,t))\equiv\Lambda^2_{(2.2)}z(x,\,t)-f\,(x,t,\,\check z(x,\,t))=0,\quad (x,\,t)\in G_h,$$
$$z(x,\,t)=\varphi(x,\,t),\quad (x,\,t)\in S_h. \tag{4.1}$$

Here $\check z(x,\,t)=z(x,t-h_t)$, $(x,t)\in\overline{G}_h$, $t>0$.

Under the condition

$$\frac{\partial}{\partial u}\,f(x,t,\,u)\le c(x,t),\quad (x,t,\,u)\in\overline{G}\times R \tag{4.2}$$

the difference scheme (4.1), (2.1) is monotone.

For simplicity we assume that the condition (4.2) is satisfied. Taking into account estimates of the solution to problem (1.1), (1.2), for the linearized difference scheme (4.1) on the special mesh (2.3) we obtain the ε-uniform estimate (similar to estimate (2.4))

$$|u(x,t)-z(x,t)|\le M\,\left[N^{-1}\ln N+N_0^{-1}\right],\quad (x,t)\in\overline{G}_h. \tag{4.3}$$

If condition (4.2) is not satisfied, then in problem (4.1), (2.1) we pass from the function $z(x,\,t)$ to the function $z^*(x,\,t)$, $z(x,\,t)=z^*(x,\,t)\,\exp(\alpha\,t)$ and choose the value α sufficiently large so that the following condition holds true

$$\frac{\partial}{\partial u}\,f(x,t,\,u)\le c(x,t)+\delta_{\overline{t}}\left[\exp(\alpha\,t)\right]p(x,t),\quad (x,t,\,u)\in\overline{G}\times R,$$

that ensures monotonicity of obtained discrete problem. Further we establish convergence of the function $z^*(x,\,t)$ to the function $u^*(x,\,t)$, $u(x,\,t)=u^*(x,\,t)\,\exp(\alpha\,t)$. Returning to the function $z(x,\,t)$, we obtain ε-uniform estimate (4.3).

5. Linearized Scheme of Improved Accuracy

In this section we present the Richardson scheme of higher accuracy, which is constructed on the base of the linearized scheme (4.1), (2.3). Let $z^i(x,t)$, $(x,t)\in\overline{G}^i_h$, $i=1,2$ be solutions of the difference schemes

$$\Lambda_{(4.1)}\left(z^i(x,\,t)\right)=0,\qquad (x,\,t)\in G^i_h, \tag{5.1a}$$
$$z^i(x,\,t)=\varphi(x,\,t),\quad (x,\,t)\in S^i_h,\quad i=1,2,$$

where $\overline{G}^i_h=\overline{G}^i_{h(3.1)}$. On the set $\overline{G}^0_h=\overline{G}^0_{h(3.1)}$ we define the function

$$z^0(x,\,t) = \gamma\, z^1(x,\,t) + (1-\gamma)\, z^2(x,\,t), \quad (x,\,t) \in \overline{G}_h^{\,0}, \tag{5.1b}$$

where $\gamma = \gamma_{(3.2)}$; $z^i(x,\,t)$, $(x,\,t) \in \overline{G}_h^{\,i}$, $i = 1, 2$ are solutions of problem (5.1), (3.1).

We call the function $z^0_{(5.1)}(x,\,t)$, $(x,\,t) \in \overline{G}_h^{\,0}$ the solution of the linearized difference scheme (5.1), (3.1), i.e. the Richardson method scheme based on the linearized scheme (4.1), (2.3).

For the solution of problem (5.1), (3.1) we obtain the estimate (similar to estimate (3.3))

$$|u(x,t) - z^0(x,t)| \le M\,\left[N^{-2}\ln^2 N + N_0^{-2}\right], \quad (x,t) \in \overline{G}_h^{0}. \tag{5.2}$$

Theorem 2. *Let hypothesis of Theorem* 1 *be satisfied. Then the solution of the linearized difference scheme* (5.1), (3.1) *converges to the solution of the boundary value problem* (1.2), (1.1) ε*-uniformly at the rate* $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-2})$.

Remark 1. Usage of the Richardson method for larger number of embedded meshes with respect to the variable t allows us to obtain schemes convergent ε-uniformly with the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-k_0})$, where $k_0 > 2$ (see, e.g., [5]). However, consideration of examples even in the case of linear problems shows that increase of the number of embedded meshes with respect to the variable x does not allow us to construct schemes convergent ε-uniformly with the accuracy in x more than 2.

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References

[1] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. High-order accurate decomposition of Richardson's method for a singularly perturbed elliptic reaction-diffusion equation. *Comp. Math. Math. Phys*, **44**(2), 309 – 316, 2004.

[2] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva. *Linear and Quasilinear Equations of Parabolic Type.* Nauka, Moscow, 1967. (in Russian)

[3] J.J.H. Miller, E. O'Riordan and G.I. Shishkin. *Fitted numerical methods for singular perturbation problems.* World Scientific, Singapore, 1996.

[4] A.A. Samarskii. *The Theory of Difference Schemes.* Marcel Dekker, New York, 2001.

[5] G.I. Shishkin. The method of increasing the accuracy of solutions of difference schemes for parabolic equations with a small parameter at the highest derivative. *USSR Comput. Maths. Math. Phys.*, **24**(6), 150 – 157, 1984.

[6] G.I. Shishkin. *Discrete approximation of singularly perturbed elliptic and parabolic equations.* Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992. (in Russian)

[7] G.I. Shishkin and L.P. Shishkina. A higher-order Richardson method for a quasilinear singularly perturbed elliptic reaction-diffusion equation. *Differential Equations*, **41**(7), 1030 – 1039, 2005.