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HIGH-ORDER ACCURATE NUMERICAL METHODS FOR PROBLEMS WITH LAYER PHENOMENA [1]

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Abstract. For singularly perturbed boundary value problems, numerical methods convergent ε-uniformly have the low accuracy. So, for parabolic convection-diffusion problem the order of convergence does not exceed one even if the problem data are sufficiently smooth. However, already for piecewise smooth initial data this order is not higher than $\frac{1}{2}$. For problems of such type, using newly developed methods such as the method based on the asymptotic expansion technique and the method of the additive splitting of singularities, we construct schemes with improved accuracy.

Key words: singular perturbation, boundary layer, parabolic convection-diffusion equation, difference scheme, parameter-uniform convergence, high-order accuracy.

1. Introduction

Problems with boundary or interior layers often appear in mathematical modelling of heat transfer and/or diffusion processes. When the layer thickness defined by a small parameter ε becomes small, then standard numerical methods give large errors. Well known special numerical methods whose errors are independent of the perturbation parameter ε, i.e. robust numerical methods, have the low order of convergence (e.g., it does not exceed one for convection-diffusion equations), that is a significant restriction to practical use.

We mention existing techniques used when constructing robust numerical methods of high-order accuracy such as well elaborated:

(a) the defect correction method,

(b) the Richardson extrapolation method,

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(c) a new method based on the asymptotic expansion technique,
(d) the method of the additive splitting of singularities.

Methods based on approaches (a), (b) are discussed in [2, 3, 4, 7, 10], approaches (c) and (d) are given in [11, 12], respectively, see the references therein. These approaches, which use the technique based on piecewise uniform meshes condensing in the layer region, allow us to construct effective numerical methods for sufficiently wide class of boundary value problems to elliptic and parabolic equations whose solutions exhibit layer phenomena.

Here, in order to demonstrate the techniques (c) and (d), we focus on singularly perturbed parabolic convection-diffusion equations with smooth and piecewise smooth initial data.

2. Problem with Smooth Data. Aim of Research

On the set $\overline{G}$, $\overline{G} = G \cup S$, $G = D \times (0, T]$, where $D = (0, d)$, we consider the boundary value problem for the parabolic equation

$$L\, u(x,t) = f(x,t), \quad (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.1}$$

Here

$$L \equiv \varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}, \quad (x,t) \in G,$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$ and S, respectively, moreover [2]

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad c_0 \le c(x,t) \le c^0, \quad p_0 \le p(x,t) \le p^0,$$
$$|f(x,t)| \le M, \quad (x,t) \in \overline{G}, \quad |\varphi(x,t)| \le M, \quad (x,t) \in S, \quad a_0, b_0, c_0, p_0 > 0;$$

the perturbation parameter ε takes arbitrary values in $(0,1]$. Assume that the data of problem (2.1) on the corner points $S_* = S_0 \cap \overline{S}^L$ satisfy the compatibility conditions which ensure the required smoothness of the solution on $\overline{G}$. Here $S = S_0 \cup S^L$, S_0 and S^L are the lower and lateral boundaries; $S_0 = \overline{S}_0$.

For small ε, a boundary layer appears in a neighbourhood of the set

$$S^l = \{(x,t):\ x = 0,\ 0 < t \le T\},$$

the left part of the lateral boundary S^L.

Let us give an ε-uniformly convergent finite difference scheme constructed by classical approximation of problem (2.1).

On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{2.2}$$

[2] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

where $\overline{\omega}$ and $\overline{\omega}_0$ are arbitrary, generally speaking, non-uniform meshes on the segments $[0, d]$ and $[0, T]$, respectively. Let $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$, and $h_t^k = t^{k+1} - t^k$, t^k, $t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. Assume that $h \le M\, N^{-1}$, $h_t \le M\, N_0^{-1}$, where $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

Problem (2.1) is approximated by the finite difference scheme [6]

$$\Lambda\, z(x,t) = f(x,t), \quad (x,t) \in G_h, \qquad z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{2.3}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}$,

$$\Lambda \equiv \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}, \quad (x,t) \in G_h,$$

$\delta_{\overline{x}\,\widehat{x}}\, z(x,t)$ is the second central difference derivative,

$$\delta_{\overline{x}\,\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)], \quad (x,t) = (x^i, t) \in G_h;$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first (forward and backward) derivatives.

Scheme (2.3), (2.2) is monotone ε-uniformly [6]. On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{2.4}$$

where $\overline{\omega}_0 = \overline{\omega}_0^u$ is a uniform mesh, $\overline{\omega}^*$ is a piecewise uniform mesh constructed as follows (see, e.g., [2, 3, 9]). The segment $[0, d]$ is divided into two parts $[0, \sigma]$ and $[\sigma, d]$, where the step-sizes are constant and equal to $h^{(1)} = 2\, \sigma\, N^{-1}$ and $h^{(2)} = 2(d - \sigma) N^{-1}$, respectively. The parameter σ is defined by

$$\sigma = \sigma(\varepsilon, N, l) = \min\left(\frac{d}{2},\ \frac{l}{m}\varepsilon \ln N\right),$$

where m is an arbitrary number from $(0, m_0)$, $m_0 = m_{(3.2)}$. Here $l = 1$; in the other piecewise-uniform meshes this parameter will be chosen later.

Theorem 1. *Let the solution $u(x,t)$ of problem* (2.1) *satisfies the estimates from Theorem* 2*, where $K = 4$. Then the difference scheme* (2.3)*,* (2.4) *converges ε-uniformly with the error estimate*

$$|u(x,t) - z(x,t)| \le M\, \left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h.$$

For the boundary value problem (2.1), our aim is to develop a finite difference scheme convergent ε-uniformly with accuracy higher than the first order.

3. A Priori Estimates and the Asymptotic Construct for Problem (2.1)

Here we give a priori estimates of the solutions and derivatives for problem (2.1) (see, e.g., [2, 3, 9]). We represent the solution of the problem as the sum of functions $u(x,t) = U(x,t) + V(x,t)$, $(x,t) \in \overline{G}$, where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution. The function $U(x,t)$, $(x,t)\in\overline{G}$ is the restriction, to $\overline{G}$, of the function $U^e(x,t)$, $(x,t)\in\overline{G}^{\,e}$, which is a solution of the "extended" problem obtained by extension of problem (2.1) beyond the left boundary S^l. The function $V(x,t)$, $(x,t)\in\overline{G}$, i.e. the boundary layer, is the solution of a homogeneous equation. Assume that

$$\frac{\partial^k}{\partial x^k}\,\varphi(x,t),\ \ \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t),\ \ \frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}f(x,t)=0,\quad (x,t)\in S_*,\ \ k,k_0\le l,\quad (3.1)$$

where $l=3n+2$, $n\ge 1$. In this case $u,U,V\in C^{l^1,\,l^1}(\overline{G})$, $l^1=n+1+\alpha$, $\alpha\in(0,1)$. For $U(x,t)$ and $V(x,t)$ we obtain the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,U(x,t)\right|\le M\,[1+\varepsilon^{n+1-k-k_0}],\quad k+k_0\le K,\qquad (3.2)$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,V(x,t)\right|\le M\,\varepsilon^{-k}\,\exp(-m\,\varepsilon^{-1}\,x),\quad (x,t)\in\overline{G},$$

where m is any number from $(0,m_0)$, $m_0=\min_{\overline{G}}[a^{-1}(x,t)\,b(x,t)]$; $K=n+1$.

Theorem 2. *Let the data of the boundary value problem* (2.1) *satisfy the condition* $a,\,b,\,c,\,p\in C^{l,\,l}(\overline{G})$, $f\in C^{l,\,l,}(\overline{G})$, $\varphi\in C^{l,\,l}(\overline{G})$, $l=3n+2$, $n\ge 1$, *and also condition* (3.1). *Then the components* $U(x,t)$, $V(x,t)$ *satisfy estimates* (3.2), *where* $K=n+1$.

We now give the problem formulation for the main term in the asymptotics of the solution that are considered in a neighbourhood of the boundary layer and outside it. We will use that problem formulation when constructing schemes for small values of the parameter. The set $\overline{G}$ can be represented in the form of a sum of two sets

$$\overline{G}=\overline{G}^{\,(1)}\cup\overline{G}^{\,(2)},\quad S^{\,(k)}=\overline{G}^{\,(k)}\setminus G^{\,(k)},\quad k=1,\,2,\qquad (3.3)$$

where $G^{\,(1)}=(0,\,\sigma)\times(0,T]$, $G^{\,(2)}=[\sigma,\,d)\times(0,T]$. The condition imposed on the parameter σ is given below in Theorem 3.

Let us introduce the function $u^2(x,t)$, $(x,t)\in\overline{G}^{\,(2)}$, i.e., two first terms in the outer asymptotic expansion of the solution of problem (2.1)

$$u^2(x,t)=u_0^2(x,t)+\varepsilon\,u_1^2(x,t),\ \ (x,t)\in\overline{G}^{\,(2)}.$$

The components $u_i^2(x,t)$ can be found by solving the following two problems

$$\begin{aligned}&L^{(1)}u_0^2(x,t)=f(x,t),\ (x,t)\in G^{\,(2)},\\&u_0^2(x,t)=\varphi(x,t),\ (x,t)\in S^{\,(2)};\end{aligned}\qquad (3.4a)$$

$$\begin{aligned}&L^{(1)}u_1^2(x,t)=-a(x,t)\frac{\partial^2}{\partial x^2}u_0^2(x,t),\ (x,t)\in G^{\,(2)},\\&u_1^2(x,t)=0,\ (x,t)\in S^{\,(2)},\end{aligned}\qquad (3.4b)$$

where

$$L^{(1)}\, u(x,t) \equiv \left\{ b(x,t)\, \frac{\partial}{\partial x} - c(x,t) - p(x,t)\, \frac{\partial}{\partial t} \right\} u(x,t).$$

We say that $u^2(x,t)$, $(x,t)\in \overline{G}^{\,(2)}$, is the solution of problem (3.4a;3.4b).

Then we find the function $u^1(x,t)$, $(x,t) \in \overline{G}^{\,(1)}$ by solving the problem

$$L_{(2.1)} u^1(x,t) = f(x,t), \quad (x,t) \in G^{\,(1)}, \tag{3.4c}$$

$$u^1(x,t) = \begin{cases} \varphi(x,t), & (x,t) \in S^{\,(1)} \cap S, \\ u^2(x,t), & (x,t) \in S^{\,(1)} \setminus S. \end{cases}$$

Theorem 3. *Let the hypothesis of Theorem 2 for $l = 4$ be fulfilled. Then under the condition $\sigma = \sigma(\varepsilon,\, \delta) = \min\left[2^{-1}\, d,\, q\, m^{-1}\, \varepsilon\, \ln(1/\delta)\right]$, where $q \geq 1$, $m = m_{(3.2)}$, δ is a sufficiently small number, the functions $u^k(x,t)$, $(x,t)\in \overline{G}^{\,(k)}$, satisfy the estimate*

$$|u(x,t) - u^k(x,t)| \leq M\, \left[\varepsilon^2 + \delta^q\right], \quad (x,t) \in \overline{G}^{\,(k)}, \quad k = 1,2.$$

4. Schemes with Improved Accuracy for Problem (2.1)

4.1. Scheme with improved convergence for finite ε

In this section we present a ε-uniformly convergent difference scheme having the rate of ε-uniform convergence of order close to two in x, for not too small values of the parameter ε, and the convergence rate of the second order in t.

On the set $\overline{G}$, we introduce the mesh (similar to (2.4))

$$\overline{G}_h = \overline{G}^{s}_h \equiv \overline{\omega}^{\,s} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}_0 = \overline{\omega}^u_0$, $\overline{\omega}^{\,s}$ is a piecewise uniform mesh on $\overline{D}$; $\overline{\omega}^{\,s} = \overline{\omega}^{\,*}_{(2.4)}(\sigma)$ for

$$\sigma = \sigma(\varepsilon, N, l = 2) = \min\left[\, 2^{-1}\, d,\;\, 2\, m^{-1}\, \varepsilon\, \ln N \,\right],\;\, m = m_{(3.2)}.$$

On the mesh $\overline{G}_h$ we consider the difference scheme

$$\Lambda^{(2)} z(x,t) = f(x,t), \quad (x,t) \in G_h, \qquad z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{4.2}$$

Here

$$\Lambda^{(2)} z(x,t) \equiv \{\Lambda_2 + \Lambda_1\}\; z(x,t),$$

$$\Lambda_2\, z(x,t) \equiv \{\varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_{\widetilde{x}}\}\, z(x,t), \quad (x,t) \in G_h \cap G^{(1)};$$

$$\Lambda_2 z(x,t) \equiv \left\{ \begin{array}{ll} \{\varepsilon a(x,t)\delta_{\overline{x}\widehat{x}} + b(x,t)\delta_{\widetilde{x}}\} z(x,t), & \text{if}\, \varepsilon N \geq M_0 \\ \{\varepsilon a(x,t)\delta_{\overline{x}\widehat{x}} + b(x,t)\delta_{x}\} z(x,t), & \text{if}\, \varepsilon N < M_0 \end{array} \right\}, (x,t) \in G_h \cap G^{(2)};$$

$$\Lambda_1\, z(x,t) \equiv \{-p(x,t)\, \delta_{\overline{t}} - c(x,t)\}\, z(x,t), \quad (x,t) \in G_h;$$

$$\overline{G}^{\,(k)} = \overline{G}^{(k)}_{(3.3)}(\sigma),\; \sigma = \sigma_{(4.1)},\; M_0 \geq 2\, d\, \max_{\overline{G}}\; \left[\, a^{-1}(x,t)\, b(x,t)\right].$$

Scheme (4.2), (4.1) is monotone ε-uniformly. For solutions of this scheme, we have the error estimate

$$|u(x,t) - z(x,t)| \le M[N^{-2}(\varepsilon + \ln^{-1} N)^{-2} + N_0^{-1}], \quad (x,t) \in \overline{G}_h, \tag{4.3}$$

when

$$\varepsilon\, N \ge M_{0(4.2)}. \tag{4.4}$$

We now give a scheme of improved convergence rate in t under condition (4.4). For this we apply an approach based on the defect correction technique (see, e.g., [2, 3]); we use scheme (4.2), (4.1) as the base scheme. We approximate problem (2.1) by the difference scheme

$$\begin{aligned} &\Lambda^{(2)} z^{(2)}(x,t) = f(x,t) + \psi^0(x,t), \quad (x,t) \in G_h, \\ &z^{(2)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{4.5}$$

Here $\Lambda^{(2)} = \Lambda^{(2)}_{(4.2)}$, $\overline{G}_h = \overline{G}_{h(4.1)}$, $\psi^0(x,t)$ is the correcting term

$$\psi^0(x,t) = \psi^0\big(x,t;\, z^{(1)}(\cdot)\big) \equiv 2^{-1}\,\tau\, p(x,t) \left\{ \begin{array}{ll} \delta_{2\,\overline{t}}\, z^{(1)}(x,t), & t \ge 2\tau \\ \dfrac{\partial^2}{\partial t^2}\, u(x,0), & t = \tau \end{array} \right\};$$

$z^{(1)}(x,t)$, $(x,t) \in \overline{G}_h$, is the solution of the scheme (4.2), (4.1), $\delta_{2\,\overline{t}}\, z(x,t)$ is the backward second-order difference derivative,

$$\delta_{2\,\overline{t}}\, z(x,t) = \left(\delta_{\overline{t}}\, z(x,t) - \delta_{\overline{t}}\, z(x,t-\tau)\right)/\tau, \quad (x,t) \in \overline{G}_h, \quad t \ge 2\tau,$$

τ is the step size of the mesh $\overline{\omega}_0$ in (4.1), $\tau = T\,N_0^{-1}$. We call the function $z^{(2)}(x,t)$, $(x,t) \in \overline{G}_h$, the solution of the scheme (4.5), (4.1), (4.4). Assume that the coefficients $a(x,t)$, $b(x,t)$ satisfy the condition

$$a(x,t) = g(x)\, b(x,t), \quad (x,t) \in \overline{G}, \tag{4.6a}$$

and the initial condition is homogeneous:

$$\varphi(x,t) = 0 \quad (x,t) \in S_0. \tag{4.6b}$$

Theorem 4. *Let condition* (4.6) *and also the estimates from Theorem* 2 *for* $K = 6$ *be fulfilled. Then the scheme* (4.5), (4.1), (4.4) *converges* ε*-uniformly. The numerical solutions under condition* (4.4) *satisfy the estimate*

$$|u(x,t) - z^{(2)}(x,t)| \le M\, \left[N^{-2}\, \ln^2 N + N_0^{-2}\right], \quad (x,t) \in \overline{G}_h.$$

4.2. Schemes based on asymptotic constructs

For small values of the parameter ε, we construct a higher-order accurate scheme by approximating problems (3.4a)–(3.4c). The mesh $\overline{G}_{h(4.1)}$ is decomposed into the sum of grid sets

$$\overline{G}_h = \overline{G}_h^{(1)} \cup \overline{G}_h^{(2)}, \quad \overline{G}_h^{(k)} = \overline{G}^{\,(k)} \cap \overline{G}_h, \quad \Gamma_h^{(k)} = \Gamma^{\,(k)} \cap \overline{G}_h, \quad k = 1, 2, \tag{4.7}$$

where $\overline{G}^{(k)} = \overline{G}^{(k)}_{(3.3)} = \overline{G}^{(k)}_{(3.3)}(\sigma),\ \sigma = \sigma_{(4.1)}$. On the grid $\overline{G}^{(2)}_h$, we approximate problem (3.4a; 3.4b) by the difference scheme

$$\Lambda^{(1)} z_0^2(x,t) = f(x,t),\ (x,t) \in G_h^{(2)},\quad z_0^2(x,t) = \varphi(x,t),\ (x,t) \in S_h^{(2)}; \tag{4.8a}$$

$$\Lambda^{(1)} z_1^2(x,t) = -a(x,t)\,\delta_{x\overline{x}}\, z_0^2(x,t) + \psi^1(x,t) + \psi^0(x,t),\ (x,t) \in G_h^{(2)}, \tag{4.8b}$$
$$z_1^2(x,t) = 0,\quad (x,t) \in S_h^{(2)}.$$

Here $\Lambda^{(1)} z(x,t) \equiv \{b(x,t)\,\delta_x - c(x,t) - p(x,t)\,\delta_{\overline{t}}\} z(x,t),\quad (x,t) \in G_h^{(2)};$

$$\psi^1(x,t) \equiv 2^{-1}\varepsilon^{-1} h^i\, b(x,t) \begin{Bmatrix} \delta_{x\overline{x}}\, z_0^2(x,t), & x > \sigma \\ \delta_{x\overline{x}}\, z_0^2(\widehat{x},t), & x = \sigma \end{Bmatrix},\quad h^i = x^{i+1} - x^i,$$

$$\psi^0(x,t) = \varepsilon^{-1}\, \psi^0_{(4.5)}\big(x,t;\, z_0^{(2)}(\cdot)\big),\quad \delta_{x\overline{x}}\, z_0^2(\widehat{x},t) = \delta_{x\overline{x}}\, z_0^2(x^{i+1},t),\ x = x^i.$$

The function $z^2(x,t) = z_0^2(x,t) + \varepsilon\, z_1^2(x,t)$, $(x,t) \in \overline{G}_h^{(2)}$, is called the solution of problem (4.8a; 4.8b). The operator $\Lambda^{(1)}$ is monotone.

For the function $z^2(x,t)$, $(x,t) \in \overline{G}_h^{(2)}$, we obtain the estimate

$$|u^2(x,t) - z^2(x,t)| \le M\,[\varepsilon^2 + N^{-2} + N_0^{-2}],\quad (x,t) \in \overline{G}_h^{(2)}. \tag{4.9}$$

On the grid $\overline{G}^{(1)}_{h(4.7)}$, problem (3.4c) is approximated by the scheme

$$\Lambda^1 z^1(x,t) \equiv \{\varepsilon\, a(x,t)\delta_{x\overline{x}} + b(x,t)\delta_{\widetilde{x}} - c(x,t) - p(x,t)\delta_{\overline{t}}\} z^1(x,t) = f(x,t),$$
$$(x,t) \in G_h^{(1)},\qquad z^1(x,t) = \begin{cases} \varphi(x,t), & (x,t) \in S_h^{(1)} \cap S, \\ z^2(x,t), & (x,t) \in S_h^{(1)} \setminus S. \end{cases} \tag{4.8c}$$

The operator Λ^1 is monotone on the grid set $G_h^{(1)}$.

For $z^1(x,t)$, $(x,t) \in \overline{G}_h^{(1)}$, we obtain the estimate

$$|u^1(x,t) - z^1(x,t)| \le M\left[\varepsilon^2 + \varepsilon\, N_0^{-1} \ln N + N^{-2}(\varepsilon + \ln^{-1} N)^{-2} + N_0^{-2}\right].$$

On the mesh $\overline{G}_{h(4.1)}$, we define the function $z(x,t)$ by the relation

$$z(x,t) = \{z^k(x,t),\quad (x,t) \in \overline{G}_h^{(k)},\quad k = 1,2\},\quad (x,t) \in \overline{G}_h,$$

where $z^k(x,t),\ (x,t) \in \overline{G}_h^{(k)}$, are the solutions of problems (4.8a; 4.8b) and (4.8c). This function $z(x,t),\ (x,t) \in \overline{G}_h$, is called the solution of the difference scheme (4.8), (4.1). For the solution of this scheme, we have the estimate

$$|u(x,t) - z(x,t)| \le M\left[\varepsilon^2 + \varepsilon\, N_0^{-1} \ln N + N^{-2}(\varepsilon + \ln^{-1} N)^{-2} + N_0^{-2}\right],$$

$(x,t) \in \overline{G}_h$. For sufficiently small values of the parameter ε (for $\varepsilon \le M\,N^{-1}$) scheme (4.8), (4.1) converges at the rate $\mathcal{O}(N^{-2} \ln^2 N + N_0^{-2})$.

The scheme with improved ε-uniform order of convergence for $\varepsilon \in (0,1]$ is constructed on the basis of schemes (4.5), (4.1) and (4.8), (4.1). For $\varepsilon \ge \varepsilon_0$, where

$$\varepsilon_0 = \varepsilon_0(N) = M^0_{(4.2)} N^{-1}, \tag{4.10}$$

we use scheme (4.5), (4.1), (4.4), while for $\varepsilon < \varepsilon_0$ we use scheme (4.8), (4.1). The function $z(x,t)$, $(x,t) \in \overline{G}_h$, constructed in this way is called the solution of the difference scheme (4.5), (4.8), (4.10), (4.1).

Theorem 5. *Let the data of the boundary value problem* (2.1) *and its solutions satisfy the hypotheses of Theorem* 4. *Then the solutions of the difference scheme* (4.5), (4.8), (4.10), (4.1) *as* N, $N_0 \to \infty$ *converges to the solution of problem* (2.1) ε*-uniformly with the estimate*

$$|u(x,t) - z(x,t)| \le M \left[N^{-2} \ln^2 N + N_0^{-2} \right], \quad (x,t) \in \overline{G}_h.$$

5. Statement of the Problem with Piecewise Smooth Initial Data. Aim of Study

In the domain $\overline{G}$ with boundary S, where $\overline{G} = G \cup S$, $G = D \times (0,T]$, $D = \{x : \ x \in (-d,\, d)\}$, we consider the parabolic equation with constant coefficients and piecewise-smooth initial data

$$L_{(5.1)}\, u(x,t) = f(x,t), \quad (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{5.1}$$

Here

$$L_{(5.1)} \equiv \varepsilon\, a \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} - c - p \frac{\partial}{\partial t}, \quad a,\ b,\ p > 0,\ c \ge 0,$$

the right-hand side $f(x,t)$ is sufficiently smooth on $\overline{G}$. The boundary function $\varphi(x,t)$ is continuous on S, sufficiently smooth on the set S^L, i.e. lateral part of S, and piecewise smooth on the set S_0, i.e. lower part of S. For $t = 0$ the first x-derivative of the function $\varphi(x,t)$ has a discontinuity of the first kind on the set $S^{(*)}$, $S^{(*)} = \{(x,t) : \ x = t = 0\}$.

By a solution of problem (2.1), we mean a function $u \in C(\overline{G}) \cap C^{\,2,1}(G)$ that satisfies the differential equation on G and the boundary condition on S. For simplicity, we assume that compatibility conditions ensuring local smoothness of the solution for fixed ε are fulfilled on the set $S_* = S_0 \cap \overline{S}^{\,L}$. The derivative $(\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^{\,*}$, where $\overline{G}^{\,*} = \overline{G} \setminus S^{\,(*)}$; for fixed ε, it is bounded on $\overline{G}^{\,*}$ and discontinuous on $S^{\,(*)}$.

We are interested in a numerical approximation to the solution $u(x,t)$, $(x,t) \in \overline{G}$. Let us specify the behaviour of the solution. Let $S^{\,\gamma} = \{(x,t) : \ x = \gamma(t),\ (x,t) \in \overline{G}\}$, and let $x = \gamma(t)$, $t \ge 0$, be the characteristic of the reduced equation passing through the point $(0,0)$. As $\varepsilon \to 0$, in the neighbourhood of the sets $S^{\,l}$ and S^{γ} there appear boundary and transient layers with typical scales ε and $\varepsilon^{1/2}$, respectively; $S^{\,l}$ is the left part of $S^{\,L}$. For simplicity, we assume that the characteristic $S^{\,\gamma}$ does not meet the boundary $S^{\,l}$, that is, the transient and boundary layers do not interact.

The classical finite difference scheme for problem (5.1) in the case of piecewise uniform meshes condensing in the layers converges ε-uniformly at the rate $\mathcal{O}(N^{-1/2}+N_0^{-1/2})$. This rate of convergence is essentially lower than that for problems with sufficiently smooth data.

Our aim is to construct a numerical method that converges ε-uniformly at the rate $\mathcal{O}(N^{-1}\ln N+N_0^{-1})$ for the problem (5.1).

Note that, when the main term of the transient layer (the function $W_1(x,t)$ in representation (6.2)) vanishes, the classical difference scheme on a piecewise uniform mesh converges ε-uniformly at the rate $\mathcal{O}(N^{-1}\ln N+N_0^{-1})$. Thus, to construct a scheme with improved convergence it is appropriate to apply the method of the additive splitting of the singularity $W_1(x,t)$, which is generated by the discontinuity in the derivative $(\partial/\partial x)\varphi(x,0)$. Such an approach is applied here to construct an improved scheme for problem (5.1).

6. A Priori Estimates for Problem (5.1)

To derive estimates for the solution of problem (5.1) and its derivatives, we apply the technique from [8, 9].

The set $\overline{G}$ is decomposed into the sum of overlapping sets

$$\overline{G}=\bigcup_j \overline{G}^{\,j},\quad j=1,\,2,\,3, \tag{6.1}$$

where

$$G^{\,1}=G^{\,1}(m^1)=\{(x,t):\ |x-\gamma(t)|<m^1,\ \ t\in(0,T]\},$$

$G^{\,1}$ and $G^{\,2}$ are the neighbourhoods of the transient and boundary layers, respectively; let $\overline{G}^{\,1}\cap\overline{G}^{\,2}=\emptyset$. We denote the solution of problems (5.1) considered on the set $\overline{G}^{\,j}$ by $u^{\,j}(x,t)$, $j=1,2,3$.

We represent the function $u^{\,1}(x,t)$, $(x,t)\in\overline{G}^{\,1}$, in the form of the sum of functions

$$u^{\,1}(x,t)=U^{\,1}(x,t)+W^{\,1}(x,t),\quad (x,t)\in\overline{G}^{\,1}, \tag{6.2}$$

where $U^{\,1}(x,t)$ and $W^{\,1}(x,t)$ are the regular and singular parts of the solutions,

$$U^{\,1}(x,t)=U^{\,1}(x,t;\,i)=U(x,t)+\sum_{k=i+1}^{K-1}W_k(x,t),\quad (x,t)\in\overline{G}^{\,1};$$

$$W^{\,1}(x,t)=W^{\,1}(x,t;\,i)=\sum_{k=1}^{i}W_k(x,t),\quad (x,t)\in\overline{G},$$

and i takes one of the values, either 1 or 2. Here $U^{\,1}(x,t)$ and $W^{\,1}(x,t)$, $(x,t)\in\overline{G}^{\,1}$, are smooth components of the solution to the inhomogeneous equation from (5.1) and the transient layer. The functions $W_k(x,t)$ are solutions of the Cauchy problems

$$L_{(5.1)}\,W_k(x,t)=0,\quad (x,t)\in I\!R\times(0,T],$$

$$W_k(x,0) = \frac{1}{2(k!)} \left[\frac{\partial^k \varphi(x,0)}{\partial x^k} \right] |x| \, x^{k-1}, \quad x \in I\!R, \; k = 1, \dots, K-1.$$

Here $\big[(\partial^k/\partial x^k)\varphi(x,0)\big] = (\partial^k/\partial x^k)\varphi(+0,0) - (\partial^k/\partial x^k)\varphi(-0,0)$.

For the components in representation (6.2), taking into account the explicit form of the functions $W_k(x,t)$, $k = 1, \dots, K-1$, we find the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, U^1(x,t) \right| \le M \big[1 + \varepsilon^{(i+1-k-k_0)/2} \rho^{i+1-k-k_0} + \varepsilon^{(i+1-k)/2} \rho^{i+1-k-2k_0} \big], \quad (x,t) \in \overline{G}^{\,1}, \tag{6.3}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, W^1(x,t) \right| \le M \big[1 + \varepsilon^{\,(1-k-k_0)/2} \, \rho^{\,1-k-k_0} + \varepsilon^{\,(1-k)/2} \, \rho^{\,1-k-2k_0} \big],$$

$$(x,t) \in \overline{G}; \quad k + 2k_0 \le \; K, \; i = 1, \, 2,$$

where $\rho = \rho(x,t;\,\varepsilon) = \varepsilon^{-1/2} \, \big| x - \gamma(t) \big| + t^{1/2}$, $\widehat{\rho} = \widehat{\rho}(\xi,t;\,\varepsilon) = \varepsilon^{-1/2} \, \big| \xi \big| + t^{1/2}$, m is an arbitrary constant.

For the function $u^{\,3}(x,t)$ we have the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, u^{\,3}(x,t) \right| \le M, \quad (x,t) \in \overline{G}^{\,3}, \quad k + 2k_0 \le K. \tag{6.4}$$

The solution $u^{\,2}(x,t)$ can be represented as the sum of functions

$$u^{\,2}(x,t) = U^{\,2}(x,t) + V^{\,2}(x,t), \quad (x,t) \in \overline{G}^{\,2}, \tag{6.5}$$

where $U^{\,2}(x,t)$ and $V^{\,2}(x,t)$ are the regular and singular parts of the solution. For the functions $U^{\,2}(x,t)$ and $V^{\,2}(x,t)$, the following estimates are valid:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, U^{\,2}(x,t) \right| \le \; M, \;\; k + 2k_0 \le K, \tag{6.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, V^{\,2}(x,t) \right| \le M \, \varepsilon^{-k} \, \exp\big(- m \, \varepsilon^{-1} \, (x + d) \big), \;\; (x,t) \in \overline{G}^{\,2},$$

where m is any constant from the interval $(0, \, m_0)$, $m_0 = a^{-1} \, b$.

Theorem 6. *Let the data of the boundary value problem* (5.1) *satisfy the condition* $f \in C^{\,l_1,\,l_1/2}(\overline{G})$, $\varphi \in C^{\,l_1}(\overline{S}_0^{\,-}) \cap C^{\,l_1}(\overline{S}_0^{\,+}) \cap C^{\,l_1/2}(\overline{S}^{\,L}) \cap C(S)$, $l_1 = l + \alpha$, *where* $l = K$, $\alpha \in (0,1)$. *Then the solution* u *and its components in representations* (6.2), (6.5) *satisfy estimates* (6.4), (6.3), (6.6).

7. Classical Finite Difference Approximations on Piecewise-Uniform Meshes. Problem (5.1)

On the set $\overline{G}$, we introduce the rectangular grid (similar to (2.2))

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{7.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the segments $[-d, d]$ and $[0, T]$. We approximate problem (5.1) by the difference scheme [6]

$$\Lambda_{(7.2)} z(x,t) = f(x,t), \ \ (x,t) \in G_h, \quad z(x,t) = \varphi(x,t), \ \ (x,t) \in S_h, \tag{7.2}$$

where

$$\Lambda_{(7.2)} \equiv \varepsilon\, a\, \delta_{\overline{x}\widehat{x}} + b\, \delta_x - c - p\, \delta_{\overline{t}}.$$

On the set $\overline{G}$, we construct a grid condensing in a neighbourhood of the boundary layer, similar to that constructed in [1, 5, 8, 9],

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{7.3}$$

where $\overline{\omega}_0 = \overline{\omega}_0^u$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $[-d,\, d]$, σ is a mesh parameter depending on ε and N. The value of σ is chosen to satisfy the condition $\sigma = \sigma(N,\, \varepsilon) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\, \ln N\,]$, where β is an arbitrary number in the half-open interval $(0, d]$, $m = m_{(6.6)}$.

Theorem 7. *Let the solution of problem* (5.1) *and its components in representations* (6.2)*,* (6.5) *satisfy estimates* (6.4)*,* (6.3)*,* (6.6) *for* $K = 4$*. Then the difference scheme* (7.2)*,* (7.3) *converges* ε*-uniformly with the estimate*

$$|\, u(x,t) - z(x,t)\,| \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h.$$

Remark 1. Let the component $W_1(x,t)$ be absent in representation (6.2), i.e., $W_1(x,t) = 0$, $(x,t) \in \overline{G}^{\,1}$. In this case we obtain the estimate

$$|u(x,t) - z(x,t)| \le M\,\left[N^{-1}\,\ln N + N_0^{-1}\right], (x,t) \in \overline{G}_h.$$

8. Improved Scheme of the Additive Splitting of Singularities for Problem (5.1)

We decompose the solution of problem (5.1) into the sum of functions $u(x,t) = u_1(x,t) + u_2(x,t)$, $(x,t) \in \overline{G}$, where $u_2(x,t) = W^1(x,t) = W^1_{(6.2)}(x,t;\, i)$, $(x,t) \in \overline{G}$, $i = 1$. The function $u_1(x,t)$ is a solution of the problem

$$L_{(5.1)} u_1(x,t) = f(x,t), \ \ (x,t) \in G, \ \ u_1(x,t) = \varphi_1(x,t), \ \ (x,t) \in S, \tag{8.1}$$

where $\varphi_1(x,t) = \varphi(x,t) - W^1(x,t)$, $(x,t) \in S$; the function $\varphi_1(x,t)$ and its first derivative in x are continuous on S_0.

We approximate problem (8.1) on mesh (7.1) by the difference scheme

$$\Lambda_{(7.2)} z_1(x,t) = f(x,t), \ \ (x,t) \in G_h, \ \ z_1(x,t) = \varphi_1(x,t), \ \ (x,t) \in S_h. \tag{8.2}$$

Using the function $z_1(x,t)$, $(x,t) \in \overline{G}_h$, we construct the function $\overline{z}_1(x,t)$, $(x,t) \in \overline{G}$, which is a piecewise bilinear interpolant. Further, we construct the function $u_0^h(x,t) = \overline{z}_1(x,t) + u_2(x,t)$, $(x,t) \in \overline{G}$. The function $u_0^h(x,t)$, $(x,t) \in \overline{G}$, is called the solution of scheme (8.2), (7.1), which is the scheme based on the method of the additive splitting of a singularity (namely, the main term of the transient layer).

Theorem 8. *Let the hypothesis of Theorem 7 be fulfilled. Then the difference scheme (8.2), (7.3) converges ε-uniformly with the error bound*

$$|u(x,t) - u_0^h(x,t)| \le M\,[N^{-1}\ln N + N_0^{-1}], \quad (x,t) \in \overline{G}.$$

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References

[1] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan and G.I. Shishkin. *Robust computational techniques for boundary layers.* Chapman and Hall/CRC Press, Boca Raton, 2000.

[2] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. $\mathcal{E}$-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems. *IMA J. Numer. Anal.*, **20**(1), 99 – 121, 2000.

[3] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. Novel defect correction high-order, in space and time, accurate schemes for parabolic singularly perturbed convection-diffusion problems. *Comput. Methods Appl. Math.*, **3**(3), 387 – 404, 2003.

[4] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. High-order accurate decomposition of richardson's method for a singularly perturbed elliptic reaction-diffusion equation. *Comp. Math. Math. Phys*, **44**(2), 309 – 316, 2004.

[5] J.J.H. Miller, E. O'Riordan and G.I. Shishkin. *Fitted numerical methods for singular perturbation problems.* World Scientific, Singapore, 1996.

[6] A.A. Samarskii. *The Theory of Difference Schemes.* Marcel Dekker, New York, 2001.

[7] G.I. Shishkin. The method of increasing the accuracy of solutions of difference schemes for parabolic equations with a small parameter at the highest derivative. *USSR Comput. Maths. Math. Phys.*, **24**(6), 150 – 157, 1984.

[8] G.I. Shishkin. Grid approximation of a singularly perturbed boundary value problem for quasilinear parabolic equations in the case of complete degeneracy in spatial variables. *Sov. J. Numer. Anal. Math. Modelling*, **6**(3), 243 – 261, 1991.

[9] G.I. Shishkin. *Discrete approximation of singularly perturbed elliptic and parabolic equations.* Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992. (in Russian)

[10] G.I. Shishkin. Finite-difference approximations of singularly perturbed elliptic equations. *Comp. Maths. Math. Phys.*, **38**(12), 1909 – 1921, 1998.

[11] G.I. Shishkin. Grid approximation of parabolic convection-diffusion equations with piecewise smooth initial data. *Dokl. Rus. Akad. Nauk.*, **405**(1), 2005. (in Russian, to appear)

[12] G.I. Shishkin. Grid approximation of the domain and solution decomposition method with improved convergence rate for singularly perturbed elliptic equations in domains with characteristic boundaries. *Comp. Maths. Math. Phys.*, **45**(7), 1155 – 1171, 2005.