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MULTISCALE PROBLEMS WITH VARIOUS BOUNDARY LAYERS FOR PDE'S IN UNBOUNDED DOMAINS[1]

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Abstract. In the first quarter plane we consider the Dirichlet problem for a singularly perturbed elliptic equation with two perturbation parameters ε_1 and ε_2 multiplying the highest derivatives of the equation and one of the first derivative respectively; $\overline{\varepsilon} = (\varepsilon_1,\, \varepsilon_2)$, $\varepsilon_1 \in (0,1]$ and $\varepsilon_2 \in [-1,1]$. For small values of the parameters boundary layers arise which may be *regular, parabolic, hyperbolic* or *elliptic*.

We construct a *formal difference scheme* (on meshes with an *infinite number of nodes*) and a *constructive difference scheme* (on meshes with a *finite number of nodes*) which converge $\overline{\varepsilon}$-uniformly, respectively, in the quarter-plane and on an arbitrary chosen bounded subdomain.

Key words: singular perturbation, various boundary layers, elliptic equation, unbounded domain, discrete approximation, ε-uniform convergence

1. Problem Formulation. Aim of the Research

In the quarter plane $\overline{D}$, where $\overline{D} = D \cup \mathit{\Gamma}$,

$$D = \{x: \; x_s \in (0,\infty), \quad s = 1,2\},$$

we consider the Dirichlet problem for the singularly perturbed equation and boundary conditions[2]

$$L_{(1.1)}\, u(x) = f(x), \quad x \in D, \quad u(x) = \varphi(x), \quad x \in \mathit{\Gamma}. \tag{1.1}$$

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[2] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that this operator (constant, grid) is introduced in formula $(j.k)$.

Here

$$L \equiv \varepsilon_1 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} + b_1(x) \frac{\partial}{\partial x_1} + \varepsilon_2\, b_2(x) \frac{\partial}{\partial x_2} - c(x),$$

the functions $a_s(x)$, $b_s(x)$, $c(x)$, $f(x)$ are assumed to be sufficiently smooth on $\overline{D}$, $s = 1, 2$, the function $\varphi(x)$ is sufficiently smooth on the sides Γ_j, $j = 1, 2$ and continuous on Γ; $\Gamma = \Gamma_1 \cup \Gamma_2$; $\Gamma_s = \overline{\Gamma}_s$, the side Γ_s is orthogonal to the axis x_s, $s = 1, 2$. We assume that the following conditions are satisfied: [3]

$$\begin{aligned} &a_0 \le a_s(x) \le a^0, \quad b_0 \le b_s(x) \le b^0, \quad c_0 \le c(x) \le c^0, \\ &a_0,\, b_0,\, c_0 > 0, \quad |f(x)| \le M, \quad x \in \overline{D}, \quad |\varphi(x)| \le M, \quad x \in \Gamma. \end{aligned}$$

The parameters ε_1 and ε_2 are components of the vector-parameter $\overline{\varepsilon}$; $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 \in (0, 1]$ and $\varepsilon_2 \in [-1, 1]$.

By the solution of the boundary value problem, we mean its classical solution, i.e., a function $u \in C^2(D) \cap C(\overline{D})$ that is bounded on $\overline{D}$ and satisfies the differential equation on D and the boundary condition on Γ. For simplicity, we suppose that the compatibility conditions ensuring the required smoothness of the solution for each fixed value of the vector-parameter $\overline{\varepsilon}$ are fulfilled on the set $\Gamma^c = \Gamma_1 \cap \Gamma_2$ of "corner points".

When the parameter ε_1 tends to zero, boundary layers arise in a neighbourhood of the boundary Γ (or its part). The nature of these layers and their properties in a neighbourhood of the sets Γ_1, Γ_2, Γ^c are determined by the vector-parameter $\overline{\varepsilon}$ (see, e.g., [4]). A boundary layer arising in a neighbourhood of the boundary Γ_2 depends on the relation between the parameters ε_1 and ε_2. This layer is regular (for $\varepsilon_1^{1/2} \ll \varepsilon_2 \le 1$), parabolic (for $|\varepsilon_2| = O(\varepsilon_1^{1/2})$) or hyperbolic (for $\varepsilon_2 < 0$, $\varepsilon_1^{1/2} \ll |\varepsilon_2| \ll 1$), or no layer appears (for $\varepsilon_2 < 0$, $|\varepsilon_2| \approx 1$). In a neighbourhood of the set Γ_1, but outside the nearest neighbourhood of Γ^c, the layer is regular (defined only by the parameter ε_1), and in a neighbourhood of the set Γ^c the layer is elliptic (for $\varepsilon_1 = o(1)$ and either $\varepsilon_2 \ge 0$ or $\varepsilon_2 < 0$, $|\varepsilon_2| = o(1)$), or the strong layer does not appear (for $\varepsilon_1 = o(1)$, $\varepsilon_2 < 0$, $|\varepsilon_2| \approx 1$).

Unlike problems in bounded domains (for singularly perturbed or regular equations), the construction of numerical methods in the case of unbounded domains is essentially complicated. The approximation of solutions of such problems on the domain of definition, as rule, requires to use discrete sets with an infinite number of mesh points. We refer to numerical methods on meshes with an infinite and finite number of nodes as *formal* and *constructive* methods, respectively. For problem (1.1), by using classical numerical approximations (see, e.g., [1]), it is not difficult to construct a formal difference scheme. It follows from Section 2 that a solution of this difference scheme on sufficiently arbitrary meshes depends on the vector-parameter $\overline{\varepsilon}$ and converges only under a very restrictive condition.

[3] Here and below M, M_i (or m^i) denote sufficiently large (or small) positive constants independent of the vector-parameter $\overline{\varepsilon}$ and the parameters of difference schemes.

To develop constructive numerical methods in the case of problems in unbounded domains it seems appropriate to use the following approach. Suppose that we are interested in finding a solution of problem (1.1) on some prescribed bounded set $\overline{D}^{\,0} \subset \overline{D}$. Let the set $\overline{D}^{\,0}$ be a rectangle defined by its lower-left and upper-right vertices $d^1 = (d^1_1,\, d^1_2)$ and $d^2 = (d^2_1,\, d^2_2)$, where d^1 is an arbitrary point of $\overline{D}$:

$$\overline{D}^{\,0} = \overline{D}^{\,0}(d^1,\, d^2), \quad \overline{D}^{\,0} = D^{\,0} \cup \Gamma^0. \tag{1.2}$$

Thus, we have $\overline{D}^{\,0} = [d^1_1,\, d^2_1] \times [d^1_2,\, d^2_2]$, $d^2 = d^1 + d^{\,0}$, $d^{\,0} = (d^0_1,\, d^0_2)$; generally speaking, the value d^0_s, that is, the size of D^0 along the x_s-axis, may depend on the parameter $\overline{\varepsilon}$; let $d^0_0 = \max[d^0_1,\, d^0_2]$. It is required to construct a numerical method that allows us to approximate the solution of problem (1.1) on the set $\overline{D}^{\,0}$. The accuracy of the discrete solution on $\overline{D}^{\,0}$ (just as the values d^0_s, $s = 1, 2$) can depend on the parameter $\overline{\varepsilon}$ and the values of N_1 and N_2, which define the number of mesh points used (in x_1 and x_2) to solve the problem numerically.

The aim of this research is to construct $\overline{\varepsilon}$-uniformly convergent formal and constructive schemes for the boundary value problem (1.1). In the case of constructive difference schemes we are interested to find the solution of problem (1.1) on the bounded set $\overline{D}^{\,0}_{(1.2)}$.

2. Formal Difference Schemes

In the case of problem (1.1) we consider formal difference schemes, viz. schemes on meshes with an infinite number of nodes.

On the set $\overline{D}$ we introduce the grid

$$\overline{D}^*_h = \overline{\omega}^*_1 \times \overline{\omega}^*_2, \tag{2.1}$$

where $\overline{\omega}^*_s$ is a mesh on $x_s \geq 0$ with arbitrarily distributed mesh points. Let $h^i_s = x^{i+1}_s - x^i_s$, $x^i_s,\, x^{i+1}_s \in \overline{\omega}^*_s$, $h_s = \max_i h^i_s$, $h = \max_s h^s$. By $N_{*s} + 1$ we denote the minimum number of nodes in $\overline{\omega}^*_s$ on a unit interval, i.e., we call N_{*s} the minimal mean (over the unit interval) density of mesh points in $\overline{\omega}^*_s$. Suppose that the condition $h \leq M N_*^{-1}$ is fulfilled, where $N_* = \min[N_{*1}, N_{*2}]$. To solve problem (1.1) we use the difference scheme [1]

$$\begin{cases} \Lambda z(x) = f(x), & x \in D^*_h, \\ z(x) = \varphi(x), & x \in \Gamma^*_h. \end{cases} \tag{2.2}$$

Here $D^*_h = D \cap \overline{D}^*_h$, $\quad \Gamma^*_h = \Gamma \cap \overline{D}^*_h$;

$$\Lambda z(x) \equiv \Big\{ \varepsilon_1 \sum_{s=1,2} a_s(x) \delta_{\overline{xs}\,\widehat{xs}} + b_1(x)\delta_{x1} + \varepsilon_2^+ b_2(x)\delta_{x2} + \varepsilon_2^- b_2(x)\delta_{\overline{x2}} - c(x) \Big\} z(x);$$

$\delta_{\overline{xs}\,\widehat{xs}} z(x)$ and $\delta_{xs} z(x)$, $\delta_{\overline{xs}} z(x)$ are the second and first (forward and backward) difference derivatives on non-uniform grids, for example:

$$\delta_{\overline{x1}\,\widehat{x1}}\, z(x) = 2\left(h_1^{i-1} + h_1^i\right)^{-1}\left(\delta_{x1} - \delta_{\overline{x1}}\right) z(x), \quad x = (x_1^i, x_2) \in D_h^*,$$

$$\varepsilon_2^+ = 2^{-1}(\varepsilon_2 + |\varepsilon_2|), \quad \varepsilon_2^- = 2^{-1}(\varepsilon_2 - |\varepsilon_2|).$$

Taking into account estimates similar to those in [4], in the case of a uniform mesh $\overline{D}_h^* = \overline{D}_h^{*u}$ we obtain the estimate

$$|u(x) - z(x)| \le M\,(\varepsilon_1 + N_*^{-1})^{-1}\, N_*^{-1}, \;\; x \in \overline{D}_h^*,$$

unimprovable with respect to N_* and $\overline{\varepsilon}$. The condition $N_*^{-1} = o(\varepsilon_1)$, $\varepsilon_1 \in (0,1]$ where N_*^{-1} is an effective stepsize, is necessary and sufficient for convergence of solutions to difference scheme (2.2) on $\overline{D}_h^{*u}$ as $N_* \to \infty$.

To construct $\overline{\varepsilon}$-uniformly convergent schemes, we use meshes condensing in a neighbourhood of the boundary layers. The rule of mesh refinement is controlled by the nature of the arising boundary layers.

On the set $\overline{D}$ we introduce the mesh

$$\overline{D}_h^* = \overline{D}_h^{*S} = \overline{\omega}_1^{*S} \times \overline{\omega}_2^{*S}, \tag{2.3a}$$

where $\overline{\omega}_s^{*S} = \overline{\omega}_s^{*S}(\sigma_s)$ is a piecewise uniform mesh on the semiaxis $x_s \ge 0$, $s = 1, 2$. The stepsizes of the mesh $\overline{\omega}_s^{*S}$ are constant on the intervals $[0, \sigma_s]$ and $[\sigma_s, \infty)$ and are equal to $h_s^{(1)} = 2\sigma_s N_{*s}^{-1}$ and $h_s^{(2)} = 2(1-\sigma_s) N_{*s}^{-1}$, respectively. The value σ_1 is chosen to satisfy the condition

$$\sigma_1 = \sigma_1(\varepsilon_1, N_{*1}) = \min\left[\, 2^{-1},\, M_1\, \varepsilon_1 \ln N_{*1} \,\right], \quad \text{where} \;\; M_1 = \frac{1}{m^1}. \tag{2.3b}$$

The magnitude of σ_2 depends on the values of ε_1, ε_2 and N_{*2} so that $\sigma_2 = \sigma_2(\varepsilon_1, \varepsilon_2, N_{*2})$:

$$\sigma_2 = \begin{cases} \min\,[2^{-1},\; M_2\, \varepsilon_1^{1/2}\, \ln N_{*2}] & \textit{for} \;\; |\varepsilon_2| \le M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\; M_3\, \varepsilon_1\, \varepsilon_2^{-1}\, \ln N_{*2}] & \textit{for} \;\; \varepsilon_2 > M^0\, \varepsilon_1^{1/2}, \\ \min\,[2^{-1},\; M_4\, |\varepsilon_2|\, \ln N_{*2}] & \textit{for} \;\; \varepsilon_2 < -M^0\, \varepsilon_1^{1/2}, \end{cases} \tag{2.3c}$$

where $M_i = \left(m^i\right)^{-1}$, $i = 2, 3, 4$. Here, by virtue of *a-priori* estimates (see, e.g., [4]) $M^0 \ge 1$ is an arbitrary number, m^i is an arbitrary number in $(0, m^{i0})$, $i = 1, \ldots, 4$. The constants m^{i0} are chosen as in [4]. The parameters ε_1, ε_2 define a width of arising boundary layers and, thus, the transition points σ_1, σ_2.

When deriving *a-priori* estimates we assume that the following condition holds:

$$a_s,\, b_s,\, c,\, f \in C^l(\overline{D}), \quad s = 1,\, 2, \quad \varphi \in C(\Gamma), \quad \varphi \in C^{l+\alpha}(\Gamma_j), \tag{2.4}$$

$$j = 1,\, 2, \quad l \ge 3K - 4, \quad K \ge 3, \quad \alpha \in (0, 1).$$

Applying the technique from [2] and taking into account *a-priori* estimates for the solutions of problem (1.1), we find the $\overline{\varepsilon}$-uniform error estimate

$$|u(x) - z(x)| \le M\, N_*^{-1} \ln N_*, \quad x \in \overline{D}_h^{*S}. \tag{2.5}$$

Theorem 1. *Let the condition* (2.4) *be satisfied, where* $K = 3$*. Then the solution of the difference scheme* (2.2), (2.3) *converges to the solution of the boundary value problem* (1.1) $\overline{\varepsilon}$*-uniformly with the estimate* (2.5)*.*

3. Constructive Difference Schemes

Suppose that it is required to develop a constructive scheme which approximates the solution of problem (1.1) on the set $\overline{D}^0_{(1.2)}$. For the set $\overline{D}^0$ we construct the extended domain

$$\overline{D}^{[0]}(\eta) = \overline{D}^{[0]}\big(\overline{D}^0;\, \eta\big). \tag{3.1}$$

Here $\overline{D}^{[0]}(\eta) = \overline{D}_\eta \cap \overline{D}$, $\overline{D}_\eta$ is the η-neighbourhood of the set $\overline{D}_{(1.2)}$, $\overline{D}_\eta = \left[d_1^1 - \eta,\, d_1^2 + \eta\right] \times \left[d_2^1 - \eta,\, d_2^2 + \eta\right]$, $d_j^i = d^i_{j(1.2)}$, $i, j = 1, 2$; $\eta \ge 1$ is an arbitrary value. Let $u^{[0]}(x)$, $x \in \overline{D}^{[0]}$, be the solution of the problem

$$Lu^{[0]}(x) = f(x), \quad x \in D^{[0]}, \tag{3.2}$$
$$u^{[0]}(x) = \varphi(x), \quad x \in \Gamma^{[0]} \cap \Gamma, \quad u^{[0]}(x) = 0, \quad x \in \Gamma^{[0]} \setminus \Gamma.$$

The functions $u^{[0]}(x)$ on the set $\Gamma^{[0]} \setminus \Gamma$ and $u(x)$ differ by a finite magnitude.

For the solution of problem (3.2) (3.1) we obtain the following estimate:

$$|u(x) - u^{[0]}(x)| \le M\, \beta, \quad x \in \overline{D}^0, \tag{3.3}$$

where $\beta = \beta(\eta) = \exp(-m\,\eta)$, $m = \min_i\, m^i$, $m^i = m^i_{(2.3)}$, $i = 1, \ldots, 4$.

For given β in the estimate (3.3), the set $\overline{D}^{[0]}$ turns out to be bounded. The condition $\eta \ge M_0\, \ln \beta^{-1}$, $M_0 = m^{-1}_{(3.3)}$ for sufficiently small β is sufficient for the set $\overline{D}^{[0]}$ to be the domain of "essential" dependence [3]. Thus, the domain of "essential" dependence, that is the domain out of which finite disturbances of the solution $u(x)$ do not influence "essentially" on the solution $u(x)$ on the set $\overline{D}^0$, is $\overline{\varepsilon}$-uniformly bounded.

To construct a constructive scheme, on the set $\overline{D}$ we introduce the set

$$\overline{D}^{\wedge} = \overline{D}^{[0]}_{(3.1)}(\eta) = \left[\widehat{d}_1^1,\, \widehat{d}_1^2\right] \times \left[\widehat{d}_2^1,\, \widehat{d}_2^2\right], \tag{3.4a}$$

where $\eta = M_0\, \ln \beta^{-1}$, and $\beta > 0$ is a sufficiently small number chosen below. On the set $\overline{D}^{\wedge}_{(3.4)}$, we introduce the mesh with an arbitrary distribution of its nodes:

$$\overline{D}_h = \overline{D}^{\wedge}_h = \overline{\omega}_1 \times\, \overline{\omega}_2, \tag{3.4b}$$

N_s+1 is the number of nodes in the mesh $\overline{\omega}_s$, $s=1,2$; let $N=\min[N_1,N_2]$. The condition $h_s \le M(d_s^0+\eta)N_s^{-1}$ is assumed to be satisfied, where h_s is the maximal stepsize of the mesh $\overline{\omega}_s$. On the mesh $\overline{D}_h$, we consider the scheme

$$\begin{cases} \Lambda z(x) = f(x), \ x \in D_h, \\ z(x) = \varphi(x), \ \ x \in \Gamma_h \cap \Gamma, \quad z(x)=0, \ \ x \in \Gamma_h \setminus \Gamma. \end{cases} \tag{3.5}$$

In order to complete the determination of the constructive difference scheme we choose the parameter β satisfying the condition

$$\beta = N^{-1}; \tag{3.4c}$$

in this case $\eta = M_0 \ln N$. The constructive difference scheme (3.5), (3.4), i.e., the scheme with the finite number of mesh points, has been thus constructed. For the solutions of the difference scheme (3.5) on the mesh $\overline{D}_h$, uniform with respect to both variables, $\overline{\omega}_s = \overline{\omega}_s^{\,u}$, $s=1,2$, we have

$$|u(x)-z(x)| \le M\big(\varepsilon_1 + (d_0^0+\ln N)N^{-1}\big)^{-1}(d_0^0+\ln N)N^{-1}, \ \ x \in \overline{D}_h^{\,0}.$$

When constructing $\overline{\varepsilon}$-uniformly convergent constructive schemes we will use, as meshes $\overline{\omega}_{s(3.4b)}$, the following meshes

$$\overline{\omega}_s^{\,u}, \ \ \overline{\omega}_s^{\,S}, \quad s=1,2, \tag{3.6a}$$

depending on the value of the vector-parameter $\overline{\varepsilon}$ and on the mutual disposition of the set $\overline{D}^{\wedge}_{(3.4)}$ and the boundary Γ. In (3.6a) $\overline{\omega}_s^{\,u}$ is a uniform mesh and $\overline{\omega}_s^{S} = \overline{\omega}_s^{S}(\sigma_s)$ is a piecewise uniform mesh on the segment $[\widehat{d}_s^1, \widehat{d}_s^2]$, $\widehat{d}_s^1 = \max[d_s^1-\eta,\, 0]$, $\widehat{d}_s^2 = d_s^2+\eta$, $d_s^i = d_{s(1.2)}^i$, $i,\, s = 1,\, 2$; σ_s is a parameter depending on $\overline{\varepsilon}$ and N_s. The stepsizes of the mesh $\overline{\omega}_s^{S}$ are constant on the segments $[\widehat{d}_s^1, \sigma_s]$ and $[\sigma_s, \widehat{d}_s^2]$ and equal to $h_s^{(1)} = 2(\sigma_s - \widehat{d}_s^1)N_s^{-1}$ and $h_s^{(2)} = 2(\widehat{d}_s^2 - \sigma_s)N_s^{-1}$, respectively. The values σ_1 and σ_2 are chosen to satisfy the conditions $\sigma_1 = \sigma_{1(2.3)}(\varepsilon_1, N_1)$, $\sigma_2 = \sigma_{2(2.3)}(\varepsilon_1,\, \varepsilon_2,\, N_2)$. The meshes $\overline{\omega}_s$ from (3.4b) are defined by the relations

$$\overline{\omega}_s = \begin{cases} \overline{\omega}_s^{\,u} \ \ for \ \ \overline{D}^{\wedge}_{(3.4)} \bigcap \Gamma_s = \emptyset, \\ \overline{\omega}_s^{\,S} \ \ for \ \ \overline{D}^{\wedge}_{(3.4)} \bigcap \Gamma_s \neq \emptyset; \qquad s=1,2. \end{cases} \tag{3.6b}$$

The difference scheme (3.5), (3.4), (3.6) for fixed values of $d_{s(1.2)}^0$, converges on $\overline{D}^{\,0}$ $\overline{\varepsilon}$-uniformly:

$$|u(x)-z(x)| \le M \sum_{s=1,2} N_s^{-1}(d_s^0 + \ln N_s), \quad x \in \overline{D}_h^{\,0}. \tag{3.7}$$

In the case of the condition d_1^0, $d_2^0 = O(\ln N)$ we have the $\overline{\varepsilon}$-uniform estimate

$$|u(x)-z(x)| \le MN^{-1}\ln N, \quad x \in \overline{D}_h^{\,0}. \tag{3.8}$$

Theorem 2. *Let the condition of Theorem* 1 *be satisfied. Then the solution of the difference scheme* (3.5)*,* (3.4)*,* (3.6) *converges on the set* $\overline{D}^0$ *to the solution of the boundary value problem* (1.1) $\bar{\varepsilon}$*-uniformly with the estimates* (3.7) *and* (3.8)*.*

Remark 1. Estimate (3.8) is similar to one in [4] for $\bar{\varepsilon}$-uniformly convergent scheme in the case of a problem in a bounded domain.

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