CHARACTERISTIC NUMBERS OF NON-AUTONOMOUS EMDEN-FOWLER TYPE EQUATIONS

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Abstract. Consider the Emden–Fowler equation $x'' = -q(t)|x|^\epsilon x$, $\epsilon > 0$, in the interval $[a, b]$. The coefficient $q(t)$ is a positive valued continuous function. The Nehari’s characteristic number $\lambda_n$ associated with the Emden–Fowler equation coincides with a minimal value of the functional $\int_a^b x^2(t) \, dt$ over all solutions of the boundary value problem

$$x'' = -q(t)|x|^\epsilon x, \quad x(a) = x(b) = 0, \quad x(t) \text{ has exactly } n \text{ zeros in } (a, b).$$

The respective solution is called by Nehari’s solution. We construct an example which shows that the Nehari’s extremal problem may have more than a unique solution.

Key words: Characteristic numbers, Emden–Fowler equation, Nehari’s solutions

1. Nehari’s Solutions

Behavior of solutions to the Emden–Fowler type equation

$$x'' = -q(t)|x|^\epsilon x, \quad \epsilon > 0,$$

where $q(t)$ is a positive valued continuous function, may be complicated if $q(t)$ is a non-monotone function.

Some regularity to the theory of the Emden–Fowler type equations of the form (1.1) is brought by the so called Nehari’s solutions.

The Nehari’s theory applies to equations of the type (1.1).
The general theorem by Nehari ([2, Theorem 3.2]) when adapted to the case under consideration states that the extremal problem below has a solution.

**Problem:**

\[ H(x) = \int_a^b \left[ x'^2 - (1 + \varepsilon)^{-1} q(t) x^{2+2\varepsilon} \right] dt \to \inf, \quad x \in \Gamma_n, \]  

(1.2)

where \( \Gamma_n \) consists of all functions \( x(t) \), which are continuous and piece-wise continuously differentiable in \([a, b]\); there exist numbers \( a_\nu \) such that

\[ a = a_0 < a_1 < \ldots < a_n = b; \]

\( x(a_0) = 0 \) and for \( \nu = 1, \ldots, n \), \( x(a_\nu) = 0 \) but \( x \neq 0 \) in \([a_{\nu-1}, a_\nu]\), and

\[ \int_{a_{\nu-1}}^{a_\nu} x'^2(t) dt = \int_{a_{\nu-1}}^{a_\nu} q(t) x^{2\varepsilon} dt. \]  

(1.3)

The respective extremal functions \( x_n(t) \) are those solutions of equation (1.1), which vanish at the points \( t = a \) and \( t = b \), have exactly \( n \) zeros in \((a, b)\) and satisfy the condition

\[ \int_a^b x'^2 dt = \int_a^b q(t) x^{2\varepsilon} dt. \]  

(1.4)

By combining (1.3) with (1.4) one gets

\[ \lambda_n(a, b) := \min_{x \in \Gamma_n} H(x) = H(x_n) = \frac{\varepsilon}{1 + \varepsilon} \int_a^b q(t) x^{2+2\varepsilon} dt = \frac{\varepsilon}{1 + \varepsilon} \int_a^b x'^2(t) dt. \]

Thus the characteristic number \( \lambda_n(a, b) \) is (up to a multiplicative constant) a minimal value of the functional \( \int_a^b x'^2(t) dt \) over the set of all solutions of the boundary value problem

\[ x'' = -q(t) x^{2\varepsilon}, \quad x(a) = x(b) = 0, \quad x(t) \text{ has } n - 1 \text{ zeros in } (a, b). \]

We will call the characteristic numbers \( \lambda_n \) by the *Nehari’s numbers* and the respective solutions of the differential equation by the *Nehari’s solutions*.

**Remark 1.** Nehari’s numbers \( \lambda_n(a, b) \) are uniquely defined by the interval \((a, b)\). In the work [2] Nehari mentioned that the theory could be developed much easier if the associated Nehari’s solution be unique. It was shown theoretically in [3] that this is not the case. There exist equations of the type (1.1), which have more than one Nehari’s solution for certain \( a, b \) and \( n \).

2. *Example: Nonuniqueness of the Nehari’s Solutions*

We construct the Emden - Fowler equation which possesses two Nehari’s solutions.
In our considerations we use systematically the lemniscatic functions sl t and cl t which can be defined as solutions of the equation $x'' = -2x^3$, subject to the initial conditions $x(0) = 0$, $x'(0) = 1$ and $x(0) = 1$, $x'(0) = 0$ respectively. Both functions are periodic with a minimal period of $4A$, where

$$A = \frac{1}{4} \int_0^1 \frac{ds}{\sqrt{1 - s^4}}.$$ One may consult the paper [1] for more properties of these functions. In many respects they behave like usual trigonometric functions.

**Equation.** Consider equation

$$x'' = -q(t) x^3, \quad t \in (-1, 1),$$

(2.1)

together with the boundary conditions

$$x(-1) = 0, \quad x(1) = 0, \quad x(t) > 0, \quad t \in (-1, 1).$$

(2.2)

The coefficient $q(t)$ is constructed as follows. Let

$$q(t) = \frac{2}{(\xi(t))^5},$$

where

$$\xi(t) = \begin{cases} \xi_1(t), & -1 \leq t \leq 0, \\ \xi_2(t), & 0 \leq t \leq 1 \end{cases}$$

and

$$\xi_1(t) = ht + \eta, \quad -1 \leq t \leq 0, \\
\xi_2(t) = -ht + \eta, \quad 0 \leq t \leq 1.$$

Thus $\xi(t)$ is a “$A$-shaped” piece-wise linear function, which depends on a positive valued parameter $h$, $\eta$ depends on $h$ as $\eta = h + 1$.

**Solutions.** Our goal: we are looking for a solution (solutions) of the problem (2.1), (2.2).

Consider two problems

$$x''_1 = -\frac{k}{(ht + \eta)^6} x^3_1, \quad x_1(-1) = 0, \quad x_1(0) = \tau, \quad x_1(t) > 0, \quad t \in (-1, 0);$$

(2.3)

$$x''_2 = -\frac{k}{(-ht + \eta)^6} x^3_2, \quad x_2(0) = \tau, \quad x_2(1) = 0, \quad x_2(t) > 0, \quad t \in (0, 1),$$

where $\tau > 0$. Let

$x_1(t)$ be a solution of the first equation of (2.3) in $[-1; 0]$;

$x_2(t)$ be a solution of the second equation of (2.3) in $[0; 1]$. 
Then the function
\[ x(t) = \begin{cases} x_1(t), & \text{if } -1 \leq t \leq 0, \\ x_2(t), & \text{if } 0 \leq t \leq 1 \end{cases} \]
is a $C^2$-solution of the problem (2.1), (2.2) if additionally the smoothness condition
\[ x_1'(0) = x_2'(0) \]
is satisfied. The problem (2.3) can be explicitly resolved as
\[ x_1(t, \beta_1) = \beta_1^\frac{1}{h} (ht + \eta) \cdot \text{sl} \left( \beta_1^\frac{1}{h} \frac{t + 1}{ht + \eta} \right), \]
where
\[ \beta_1 = x_1'(-1) > 0 \]
and
\[ x_1(0; \beta_1) = \tau. \]
The derivative is given by
\[ x_1'(t; \beta_1) = \beta_1^\frac{1}{h} h \cdot \text{sl} \left( \beta_1^\frac{1}{h} \frac{t + 1}{ht + \eta} \right) + \beta_1 \frac{-h + \eta}{ht + \eta} \cdot \text{sl}' \left( \beta_1^\frac{1}{h} \frac{t + 1}{ht + \eta} \right). \]
Similar formulas are valid for $x_2(t)$. Notice that $x_2'(1) = -\beta_2 < 0$. In order to get an explicit formula for a solution of the BVP (2.1), (2.2) one have to solve a system of two equations with respect to $(\beta_1, \beta_2)$
\[ x_1(0; \beta_1) = x_2(0; \beta_2), \quad x_1'(0; \beta_1) = x_2'(0; \beta_2). \]
This system after replacements and simplifications looks as
\[ \begin{cases} \beta_1^\frac{1}{h} \cdot \text{sl} \left( \frac{\beta_1}{\eta} \right) = \beta_2^\frac{1}{h} \cdot \text{sl} \left( \frac{\beta_2}{\eta} \right), \\ \beta_1^\frac{1}{h} h \cdot \text{sl} \left( \frac{\beta_1}{\eta} \right) + \beta_1 \cdot \text{sl}' \left( \frac{\beta_1}{\eta} \right) = -\beta_2^\frac{1}{h} h \cdot \text{sl} \left( \frac{\beta_2}{\eta} \right) - \beta_2 \cdot \text{sl}' \left( \frac{\beta_2}{\eta} \right), \end{cases} \]
where $0 < \frac{\beta_1}{\eta}, \frac{\beta_2}{\eta} < 2A$. In new variables $u := \frac{\beta_1}{\eta}, \ v := \frac{\beta_2}{\eta}$ the system takes the form
\[ \begin{cases} u \cdot \text{sl} u = v \cdot \text{sl} v, & 0 < u, v < 2A, \\ h u \cdot \text{sl} u + u^2 \cdot \text{sl}' u = -h v \cdot \text{sl} v - v^2 \cdot \text{sl}' v, & h > 0. \end{cases} \tag{2.4} \]
Notice that if a solution $(\bar{u}, \bar{v})$ of the system (2.4) exists, then a solution $x(t)$ of the BVP (2.1), (2.2) can be constructed such that
\[ x'(-1) = \beta_1 = \bar{u}^2 (h + 1)^2, \quad x'(1) = -\beta_2 = -\bar{v}^2 (h + 1)^2. \]
Proposition 1. For $h$ large the system (2.4) has exactly three solutions, which have the following characteristics.

1. There exists a unique symmetric solution $(u_0, v_0)$, that is, $u_0 = v_0$. One has that $(u_0, v_0) \to (2A, 2A)$ as $h \to +\infty$.

2. There exists a unique solution $(u_1, v_1)$ in the triangle $\{0 < u, v < 2A, v > u\}$ for $h$ large. Moreover, $(u_1, v_1) \to (0, 2A)$ as $h \to +\infty$.

3. There exists a unique solution $(u_2, v_2)$ in the triangle $\{0 < u, v < 2A, v < u\}$ for $h$ large. Solutions $(u_1, v_1)$ and $(u_2, v_2)$ are symmetric, that is, $(v_2, u_2) = (u_1, v_1)$.

Zeros of the functions $\Phi(u, v) = u \, s \, u - v \, s \, v$ and $\Psi(u, v) = hu \, s \, u + u^2 \, s \, u' + hv \, s \, v + v^2 \, s \, v'$ in the square $Q = \{(u, v): 0 \leq u, v \leq 2A\}$ for $h > 1$ are depicted in the Figure 1. Notice that a set of zeros of $\Phi$ consists of the diagonal $u = v$ and two symmetric branches.

![Figure 1. Zeros of $\Phi(u, v)$ (solid line) and $\Psi(u, v)$ (dashed line), $h = 2$.](image)

Nehari’s numbers. The respective solutions of the boundary value problem (2.1), (2.2) look as shown in the picture.

![Nehari’s numbers](image)

Let $H = \frac{1}{2} \int_{-1}^{1} x^2(t) \, dt$. Denote by $H_{\text{sym}}$ and $H_{\text{asym}}$ the respective values of $H$ for a symmetric solution (which is depicted by solid line), and for asymmetric solutions (depicted by dashed lines). Notice that $H(x)$ is the same for both asymmetric solutions. The values of $H_{\text{sym}}$ and $H_{\text{asym}}$ are
\[ H_{\text{sym}} = \frac{2}{3} u_0 \left( \frac{u_1}{h+1} \right) - \text{sl} \left( \frac{u_1}{h+1} \right) \text{sl} \left( \frac{u_1}{h+1} \right), \]
\[ H_{\text{asym}} = \frac{1}{3} u_1^2 \left( \frac{u_1}{h+1} \right) - \text{sl} \left( \frac{u_1}{h+1} \right) \text{sl} \left( \frac{u_1}{h+1} \right) \]
\[ + \frac{1}{3} u_1^2 \left( \frac{u_1}{h+1} \right) \text{sl} \left( \frac{u_1}{h+1} \right). \]

Proposition 2.

\[ \frac{H_{\text{sym}}}{H_{\text{asym}}} \xrightarrow{h \to +\infty} 2. \]

Therefore for \( h \) large two asymmetric solutions are the Nehari’s solutions.

References

