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DIRICHLET TYPE PROBLEM FOR THE SYSTEM OF ELLIPTIC EQUATIONS, WHICH ORDER DEGENERATE AT A LINE

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Abstract. Dirichlet type problem in bonded domain for the system of linear elliptic equations of second order, which degenerate into first order system at a line crossing the domain, is studied. The existence and uniqueness of a solution of this problem in the Höllder class of functions are proved without any additional condition at line of degeneracy. The only requirement is that the solution is bounded.
Key words: elliptic equations, Dirichlet problem

1. Statement of the Problem

Let $D \in \mathbb{R}^{n+1}$, $n \geq 1$, $\partial D = \Gamma$, be a bounded domain of points $x = (x_0, x_1, \ldots, x_n)$ such that D contains the cylinder

$$C_R := \{(x_0, x') : \ x' = (x_1, ..., x_n) \in \mathbb{R}^n, \ |x'| < R, \ 0 < x_0 < h\},$$

both bases of which lie on Γ. Thus, line $x' = 0$ is the axis of cylinder C_R and it cross the domain D intersecting with Γ at two points $P_1\,(0,0)$, $P_2\,(h,0)$.

We consider the equation

$$\mathcal{L}u := \sum_{i,j=0}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=0}^{n} b_i(x) u_{x_i} + c\,(x)\,u = f\,(x)\,, \ x \in D, \tag{1.1}$$

assuming that the coefficients and right-hand side are bounded in D, $a_{ij} = a_{ji}$ $\left(i, j = \overline{1, n}\right)$.

We use below the following notation: Ω is projection of D onto the plane $x_0 = 0$, $S = \partial \Omega$, $\Omega_0 = \Omega \setminus \{x' = 0\}$, $r = |x'|$, $d = \max_{x' \in \overline{\Omega}} r$, $|\cdot|_{l;D}$ and $|\cdot|_{l,\alpha;D}$ are the norms in Banach spaces $C^l\left(\overline{D}\right)$ and $C^{l;\alpha}\left(\overline{D}\right)$, respectively, $l \geq 0$ is an integer,

$$C_\rho = \{x:\ r < \rho,\ 0 < x_0 < h\}, \quad C_\rho^0 = C_\rho \setminus \{x' = 0\},$$

$$Q_\rho = \{x:\ r = \rho,\ 0 \le x_0 \le h\},\ D_\delta = D \setminus \{x:\ r \le \delta\},\ \Gamma_\delta = \Gamma \setminus \{x:\ r \le \delta\},$$

where $0 \le \delta \le R$. We assume, that $c(x) < 0$ in D_0 and that there exists the continuous in $\overline{\Omega}$ functions a_i $(i = 1, 2)$ such that $0 < a_1(x') \le a_2(x')$ in $\Omega_0 \cup S$, $a_2(x') \to 0$ as $x' \to 0$, and suppose that the main part of operator $\mathcal{L}$ satisfies the inequality

$$a_1(x')\,|\xi|^2 \le \sum_{i,j=0}^{n} a_{ij}(x)\xi_i\xi_j \le a_2(x')\,|\xi|^2 \quad \forall x \in D \text{ and } \forall\, \xi \in \mathbb{R}^{n+1}. \tag{1.2}$$

Therefore, according to (1.2), equation (1.1) is elliptic in D_0 and its order degenerate at the line $x' = 0$.

Let us to define the class of functions

$$C_{loc}^{l,\alpha}(D_0) := \left\{u:\ u \in C^{l,\alpha}\left(\overline{D_\delta}\right)\ \forall \delta > 0,\ |u| < \infty \text{ in } D_0\right\},$$

where $l \ge 0$ is an integer.

We study the following problem:

$$\begin{cases} \mathcal{L}u = f \text{ in } D_0, & u \in C_{loc}^{2,\alpha}(D_0), \\ u\,|_{\Gamma_0} = \varphi, \end{cases} \tag{1.3}$$

where φ is a given function.

Note, that problem (1.3) in the case where the operator $\mathcal{L}$ degenerates at line $x' = 0$ into ultraparabolic one, is discussed in [2, 4].

2. Auxilaries

Here we discuss the properties of operator $\mathcal{L}$ that will imply the uniqueness of the solution of problem (1.3).

Lemma 1. *Let $u \in C^2(D) \cap C(D_0 \cup \Gamma_0)$ be a solution of equation $Lu = 0$ and let there exists a positive in $\Omega_0 \cup S$ function ω satisfying the following conditions:*

$$\omega(x') \to +\infty \text{ uniformly as } x' \to 0, \tag{2.1}$$

$$L\omega < 0 \text{ in } D_0 \ \left(k = \overline{1, m}\right). \tag{2.2}$$

If u is uniformly bounded in D_0 and $u\,|_{\Gamma_0} = 0$, then $u \equiv 0$ in D_0.

Proof. Introduce the function v by the formula

$$u = \omega v \tag{7}$$

Let $\varepsilon > 0$ be arbitrary. Since u is uniformly bounded in D_0, due to (2.1) there exits cylinder C_{r_ε} such that

$$\left|v\right| = \omega^{-1}\left|u\right| < \varepsilon \text{ in } C^0_{r_\varepsilon} \cup \Gamma_{r_\varepsilon}. \tag{8}$$

We will show that this inequality holds also in D_{r_ε}. Putting (7) into equation $\mathcal{L}u=0$, we obtain that v satisfies the equation

$$\tilde{\mathcal{L}}v := \sum_{i,j=0}^{n} a_{ij}(x)v_{x_i x_j} + \sum_{i=0}^{n} \tilde{b}_i(x)v_{x_i} + \tilde{c}\,(x)\,v = 0,$$

where $\tilde{b}_i = b_i + 2\sum_{j=0}^{n} a_{ij}\omega_{x_j}$ $\left(i = \overline{0,n}\right)$ and $\tilde{c} = \omega^{-1}L\omega$. So we have $\tilde{c}\,(x) < 0$ in D_0 because of (2.2). Therefore, v cannot attain in D_0 neither positive maximum or negative minimum. Let us note, that $\partial D_{r_\varepsilon} = \Gamma_{r_\varepsilon} \cup Q_{r_\varepsilon}$, $v\left|_{\Gamma_{r_\varepsilon}} = 0\right.$ and $|v| < \varepsilon$ on Q_{r_ε}, i.e., the inequality $|v| < \varepsilon$ holds in D_{r_ε}. Hence, it holds due to (8) everywhere in D_0. Consequently, we have $u = \omega^{-1}v \equiv 0$ in D_0, because ε is arbitrarily chosen. ∎

Let us give the properties of operator $\mathcal{L}$, under which function

$$\omega(r) = K - \ln r$$

with large enough constant K satisfies the conditions of Lemma 1.

Obviously, $\omega\,(r) \to +\infty$ as $x' \to 0$. Here and in following we will assume that $K > e^d$. Then $\omega\,(r) > 0$ $\forall x' \in \Omega_0 \cup S$, evidently. (We recall that $d = \max_{x\in\overline{\Omega}} r$).

Lemma 2. *Let there exists* $\sup_{D_0} c = -c_0 < 0$ *and let one of the following conditions be fulfilled:*

a) $a_2\,(x') = O\,(r^{\mu})$ *in* Ω, *where* μ *is any positive number, and there exist a number* ν, $0 \le \nu < \mu$, *and cylinder* $C^0_\rho \subset D_0$ *such that*

$$\inf_{C^0_\rho} r^{-\nu} \sum_{i=1}^{n} x_i b_i\,(x) > 0 \ \text{ for any } \rho > 0;$$

b) $a_2\,(x') = O\,(r^{\mu})$ *in* Ω *with any* $\mu \ge 2$, $b_i\,(x) = O\,(r^{\nu})$ *uniformly in* D *with any* $\nu \ge 1$.

If K *is large enough, then* $\omega(r) = K - \ln r$ *satisfy in* D_0 *the inequality* $L\omega < 0$.

Proof. Consider operator $L_0 = L - c$. By direct calculation we obtain that

$$L_0\omega = r^{-2}\left(2r^{-2}\sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - \sum_{i=1}^{n}\Big(a_{ii}(x) + x_i b_i\,(x)\Big)\right).$$

Note, that in view of (1.2) the inequalities

$$\sum_{i,j=1}^{n} a_{ij}(x)x_i x_j \ge a_2\,(x')\,r^2, \quad \sum_{i=1}^{n} a_{ii}(x) > 0$$

are valid for each $x \in D_0$. Taking those in account, we get that

$$L_0\omega < r^{-2}\psi(x) \quad \forall x \in D_0,$$

where $\psi(x) = 2a_2(x') - \sum_{i=1}^n x_i b_i(x)$.

Let condition a) be fulfilled and let

$$\inf_{C_\rho^0} r^{-\nu} \sum_{i=1}^n x_i b_i(x) = \beta.$$

Obviously, then the inequality $\psi(x) \le -\beta + 2r^{-\nu}a_2(x')$ holds for every $x \in C_\rho^0$. Since $a_2(x') = O(r^\mu)$ in Ω and $\mu > \nu$, we obtain that $\psi(x)$ is negative in some cylinder $C_{r_0}^0 \subseteq C_\rho^0$ with small enough r_0, because β is positive according to assumption of lemma. Let $\overline{\psi}_0 = \sup_{\overline{D}_{r_0}} |\psi|$, then

$$L_0\omega \le r^{-2}\psi(x) \le r_0^{-2}\overline{\psi}_0 \quad x \in D_0.$$

Thus, we get that if $K > c_0^{-1}r_0^{-2}\overline{\psi}_0 + \ln d$, then, evidently,

$$L\omega = (L_0 + c)\,\omega \le r^{-2}\psi(x) + c(K - \ln r) \le r_0^{-2}\overline{\psi}_0 - c_0K + c_0\ln r < 0$$

in domain D_0.

Now assume that conditions b) hold. Let $\gamma = \min\{\mu,\ \nu\}$. In such a case the relation $\psi(x) = O\left(r^{\gamma+2}\right)$ holds uniformly in D_0. Hence, $r^{-2}\psi(x)$ is uniformly bounded in D_0. Let $\kappa = \sup_{D_0} r^{-2}|\psi|$. Then we obtain, that

$$L\omega \le r^{-2}\psi(x) + c(K - \ln r) \le \kappa - c_0K + c_0\ln r \text{ in } D_0.$$

It follows from here that $L\omega < 0$, if $K > c_0^{-1}\kappa + \ln d$. Hence, if K is suitably chosen, both conditions a) and b) imply the inequality $L\omega < 0$ in D_0. ■

3. The Existence and Uniqueness of the Solution of Problem (1.3)

We shall prove the existence of the solution of problem (1.3) in the class of functions $C_{loc}^{2,\alpha}(D_0)$ defined above. Let us assume that

$$a_{ij},\ b_i,\ c \text{ and } f \in C_{loc}^{0,\alpha}(D_0),\ \Gamma \in C^{2,\alpha},\ \varphi \in C^{2,\alpha}(\overline{D}),\ 0 < \alpha \le 1. \qquad (9)$$

Note, that domain D_δ participating in definition of $C_{loc}^{2,\alpha}(D_0)$ is not smooth, because it has two edges $\{r = \delta,\ x_0 = 0\}$ and $\{r = \delta,\ x_0 = h\}$, which are in fact the spheres of dimension $n-1$.

Let us take the domain D_δ^* with the boundary $\Gamma_\delta^* \in C^{2,\alpha}$ such that $D_0 \subset D_\delta \subset D_\delta^*$. We may chose D_δ^* so that a part of boundary Γ_δ^* coincides with surface $\Gamma_{2\delta}$ and lateral surface Q_δ of cylinder $\{r \le \delta,\ \delta \le x_0 \le h - \delta\}$. The remaining part of Γ_δ^* lies in the cylinder C_δ. It consists from two surfaces $\sigma_\delta^{(1)}$ and $\sigma_\delta^{(2)}$:

- $\sigma_\delta^{(1)}$ joining spheres $\{r = 2\delta,\ x_0 = 0\}$, and $\{r = \delta,\ x_0 = \delta\}$,

- $\sigma_\delta^{(2)}$ joining spheres $\{r = 2\delta,\ x_0 = h\}$ and $\{r = \delta,\ x_0 = h - \delta\}$.

Let $\left\{D^*_{\delta_k}\right\}$ be the sequence of domains $D^*_{\delta_k} \subset D^*_{\delta_{k+1}}$, $\lim_{k\to\infty} \delta_k = 0$, which are constructed by the rule given above. Observe, that accordingly to condition (1.2) operator $\mathcal{L}$ does not degenerate in any domain $D^*_{\delta_k}$. Moreover, it is uniformly elliptic in each $D^*_{\delta_k}$, $k = 1, 2, ...$ Therefore, there exists unique solution $u_k \in C^{2,\alpha}\left(\overline{D^*_{\delta_k}}\right)$ of Dirichlet problem

$$\mathcal{L}u = f \text{ in } D^*_{\delta_k},\ u\Big|_{\partial D^*_{\delta_k}} = \varphi,$$

and the following estimate holds (see [1, 3])

$$|u_k|_{0;D^*_{\delta_k}} \le M, \tag{10}$$

where $M = \max\left\{|\varphi|_{0;D}\,;\ \left|fc^{-1}\right|_{0;D}\right\}$.

Let us introduce the sequence $\left\{\tilde{u}^{(k)}\right\}$ by formula

$$\tilde{u}_k(x) = \begin{cases} u_k(x),\ x \in \overline{D^*_{\delta_k}}; \\ \varphi(x),\ x \in D \backslash \overline{D^*_{\delta_k}}. \end{cases}$$

It is easy to see, that every term $\tilde{u}_k$ of this sequence is defined in $\overline{D}$ and $\tilde{u}_k \in C^{2,\alpha}\left(\overline{D}\right)$, $k = 1, 2, ...$

Lemma 3. *There exists a subsequence of* $\{\tilde{u}_k\}$ *strongly convergent in the space* $C^{2,\alpha}\left(\overline{D_\varepsilon}\right)$*, where* ε *is arbitrarily chosen.*

Proof. Let ε be arbitrary. Applying to operator $\mathcal{L}$ the a priori estimates inclusively to the part Γ_ε of boundary ∂D_ε [3] we get that

$$|\tilde{u}_k|_{2,\alpha;D_\varepsilon} \le N_\varepsilon\left(|\mathcal{L}\tilde{u}_k|_{0,\alpha;D^*_\varepsilon} + |\tilde{u}_k|_{0,\alpha;D^*_\varepsilon} + |\varphi|_{2,\alpha;D^*_\varepsilon}\right)$$

with a constant N_ε depending on ε. Due to obvious estimates

$$|\mathcal{L}\tilde{u}_k|_{0,\alpha;D^*_\varepsilon} \le M_1 = \max\left\{|\mathcal{L}\varphi|_{0,\alpha;D},\ |f|_{0,\alpha;D}\right\},$$
$$|\tilde{u}_k|_{0,\alpha;D^*_\varepsilon} \le M,\ \ |\varphi|_{2,\alpha;D^*_\varepsilon} \le M_2 = |\varphi|_{2,\alpha;D}$$

we obtain, that

$$|\tilde{u}_k|_{2,\alpha;D_\varepsilon} \le N_\varepsilon\left(M + M_1 + M_2\right).$$

Thus, $\{\tilde{u}_k\}$ is compact in $C^{2,\alpha}\left(\overline{D_\varepsilon}\right)$. This yields the existence of a subsequence strongly convergent in the space $C^{2,\alpha}\left(\overline{D_\varepsilon}\right)$. ■

Remark 1. Assume, that subsequence $\{\tilde{u}^\varepsilon_k\} \subset \{\tilde{u}_k\}$ strongly converges in $C^{2,\alpha}\left(\overline{D_\varepsilon}\right)$ to u_ε. Then $\mathcal{L}u_\varepsilon = f$ in D_ε, $u|_{\Gamma_\varepsilon} = \varphi$, evidently, and in view of (10) there holds the estimate $|u_\varepsilon|_{0;D_\varepsilon} \le M$.

Using the diagonalisation method we shall show, that one can choose the subsequence of sequence $\{\tilde{u}_k\}$ converges to the solution of problem (1.3). Let $\{\varepsilon_k\}$ be a vanishing sequence of positive numbers and let $\{D_{\varepsilon_k}\}$ be sequence of the corresponding domains. According to Lemma 3 there exist subsequences $\{\tilde{u}_k^{(i)}\}$ strongly convergent in $C^2(\overline{D_{\varepsilon_{k_i}}})$, $i = 1, 2, ...$, and such that $\{\tilde{u}_k^{(i+1)}\} \subset \{\tilde{u}_k^{(i)}\} \subset \{\tilde{u}_k\}$ for all i. Let us consider the sequence $\{v_k\}$, $v_k = \tilde{u}_k^{(k)}$.

Theorem 1. *Let the smoothness conditions (9) be fulfilled and let $c < 0$ in $\overline{D}$. If either condition a) or b) of Lemma 2 is satisfied, then the sequence $\{v_k\}$ determined above converges to the solution u of problem (1.3). This solution is unique.*

Proof. Let δ be arbitrarily chosen and let k_0 be such that $\varepsilon_k \le \delta\ \forall k > k_0$. Then $D_\delta \subset D_{\varepsilon_{k_0}}$ and $v_k \in \{\tilde{u}_k^{(k_0)}\}$ for all $k > k_0$. Thus, sequence $\{v_k\}$ strongly converges in the space $C^2(\overline{D_{\varepsilon_{k_0}}})$ to some limit u because of the choice of $\{\tilde{u}_k^{(k_0)}\}$. Therefore, since $C^2(\overline{D_\delta}) \subset C^2(\overline{D_{\varepsilon_{k_0}}})$, $|v_k - u|_{2;D_\delta} \to 0$, $k \to \infty$. Taking in account remark 1, we obtain that

$$\mathcal{L}u = f \text{ in } D_\delta, \quad u\,|_{\Gamma_\delta} = \varphi$$

and $|u|_{0;D_\delta} \le M$. Furthermore, $u \in C^{2,\alpha}\left(\overline{D_\delta}\right)$, because $C^{2,\alpha}\left(\overline{D_\delta}\right)$ is complete space. Since δ is arbitrary chosen, we get that

$$\mathcal{L}u = f \ \text{ in } D_0, \quad u\,|_{\Gamma_0} = \varphi$$

and $|u| \le M$ in D_0, i.e., u is the solution of problem (1.3).

The uniqueness of solution u follows from Lemma 2. Indeed, if $u^{(1)}$ and $u^{(2)}$ are two solutions of problem (1.3), then $\left(u^{(1)} - u^{(2)}\right)|_{\Gamma_0} = 0$. Since operator $\mathcal{L}$ satisfies either condition a) or b) of Lemma 2, it follows from this lemma that $u^{(1)} - u^{(2)} \equiv 0$ in D_0. ■

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