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WAVELET ANALYSIS FOR EFFECTIVE PROPERTIES OF DOUBLY PERIODIC COMPOSITE MATERIALS

C. CATTANI[1], E. LASERRA[1] and S. ROGOSIN[2]

[1] *University of Salerno*

Via Ponte Don Melillo, I-84084 Fisciano (Sa), Italy

E-mail: ccattani@unisa.it

[2] *Belarusian State University*

Nezavisimosti ave 4, BY-220050 Minsk, Belarus

E-mail: rogosin@bsu.by

Abstract. We consider 2D composite materials with doubly periodic cells and randomly distributed inclusions in the representative cell. The wavelet techniques is used in order to approximate the effective characteristics of such materials (e.g. thermo- and electric-conductivity, effective permeability etc.) It is applied to the functional equations arising.

Key words: composite material, doubly periodic structure, wavelets, functional equations.

1. Introduction

Wavelet analysis becomes nowadays very important interdisciplinary subject which unifies the results of Mathematical Analysis, Linear and Nonlinear Algebra, Differential and Difference Equations, Numerical Analysis, Computer Science etc. Among the applications we have to point out those to the study of different properties of composite materials (see e.g. [3] and references therein). The standard approach to the study of the effective properties of composite materials with the periodic structure (i.e. the properties of the assembled material as a whole) is the homogenization methods highly developed in the recent two decades (see, *e.g.*, [1, 5, 12]).

Usually the homogenization technique is used for materials with the rich microstructure but it leads to the property that the composite material on the consideration is (almost) of laminate type (see [11]). Recently a new model for composite materials with a rich microstructure was proposed (see [2, 6, 7]).

It is supposed that the material itself consists of doubly periodically situated cells in which the inclusions are spread randomly in a sense randomly. For such composites the effective characteristics are determined in the form of the infinite series containing so called Eisenstein's sums (see [10]). Since these analytic representations are too compound we propose in this paper to use a family of 2D (complex) wavelets in order to approximate the effective parameters of the above said composite materials. Due to periodic nature of the global structure of the material we propose here a periodic variant of compactly supported wavelets (as an analog of those one-dimensional periodic wavelets discussed in [4]). We exploit here the ideas from [8] for construction of the corresponding family of wavelets.

2. Effective Conductivity of Doubly Periodic Composite Materials

Let us consider a lattice L which is defined by the two fundamental translation vectors 1 and $\imath$ ($\imath^2 = -1$) in the plane $\mathbb{C} \cong \mathbb{R}^2$ of the complex variable $z = x + \imath y$. The *fundamental cell* $Q_{(0,0)}$ is the square

$$\{ z = t_1 + \imath t_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2}, \, p = 1, 2 \}.$$

Let $\mathcal{E} := \bigcup\limits_{m_1, m_2} \{m_1 + \imath m_2\}$ be the set of the *lattice points*, where $m_1, m_2 \in \mathbb{Z}$. Let us denote the cells corresponding to the points of the lattice $\mathcal{E}$ as

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \{z \in \mathbb{C} : \, z - m_1 - \imath m_2 \in Q_{(0,0)}\}.$$

Let us consider the situation when mutually disjoint equal disks (inclusions) $D_k := \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, ..., N)$ are located inside of fundamental cell $Q_{(0,0)}$, and periodically repeated in all cells $Q_{(m_1,m_2)}$. Let us denote by $T_k := \{z \in \mathbb{C} : |z - a_k| = r\}$ the boundary of the corresponding inclusion and consider the connected domain $D_0 := Q_{(0,0)} \setminus (\bigcup\limits_{k=1}^{N} D_k \cup T_k)$ obtained by removing of the inclusions from fundamental cell $Q_{(0,0)}$.

The problem is to define the effective conductivity of the doubly periodic composite material with matrix $\mathcal{D}_{per} = \bigcup\limits_{m_1,m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2)$ and inclusions $\mathcal{D}_{inc} = \bigcup\limits_{m_1,m_2} \bigcup\limits_{k=1}^{N} (D_k + m_1 + \imath m_2)$ occupied by materials of conductivities $\lambda = 1$ and $\lambda_1 > 0$, respectively. This problem is equivalent to determination of the potential of corresponding field, i.e., of finding a function $u(z)$ satisfying the Laplace equation in each component of the composite material:

$$\triangle u = 0, \qquad z \in \mathcal{D}_{per} \bigcup \mathcal{D}_{inc}, \tag{2.1}$$

and conjugation conditions:

$$u^+(t) = u^-(t), \quad \frac{\partial u^+}{\partial n}(t) = \lambda_1 \frac{\partial u^-}{\partial n}(t), \ t \in \bigcup_{m_1, m_2} T_k, \ \ k = 1, 2, ..., N, \quad (2.2)$$

where $\dfrac{\partial}{\partial n}$ is the outward normal derivative and

$$u^+(t) := \lim_{z \to t, z \in D_0} U(z), \ u^-(t) := \lim_{z \to t, z \in D_k} U(z), \ t \in \bigcup_{m_1, m_2} T_k, \ k = 1, 2, ..., N.$$

We also impose the quasi periodicity conditions on $u(z)$, i.e., $u(z)$ has constant jump in the direction of each fundamental translation vector:

$$u(z+1) = u(z) + 1, \quad u(z+\imath) = u(z). \quad (2.3)$$

Let us introduce the complex potentials $\varphi(z)$, $\varphi_k(z)$ analytic in D_0 and D_k, respectively, and continuously differentiable in the closures of D_0 and D_k. The harmonic potential and complex analytic potentials φ and φ_k are related by the equalities

$$u(z) = \begin{cases} \Re\left(\varphi(z) + z\right), \ z \in D_0, \\ \frac{2}{1+\lambda_1} \Re \varphi_k(z), \ z \in D_k, \ \ k = 1, 2, ...N. \end{cases} \quad (2.4)$$

The real conditions (2.2) can be rewritten following in terms of the complex potentials (see [6])

$$\varphi(t) = \varphi_k(t) - \rho \ \overline{\varphi_k(t)} - t, \quad |t - a_k| = r, \ \rho = \frac{1 - \lambda_1}{1 + \lambda_1}, \quad k = 1, 2, ..., N. \quad (2.5)$$

The problem (2.5) is a particular case of so called $\mathbb{R}$-linear conjugation problem.

To determine the current $\nabla u(x, y)$ we need to introduce another complex potentials which are simply derivatives of the potentials (2.4):

$$\begin{aligned} \psi(z) &:= \frac{\partial \varphi}{\partial z} = \frac{\partial u}{\partial x} - \imath \frac{\partial u}{\partial y}, \quad z \in D_0 \\ \psi_k(z) &:= \frac{\partial \varphi_k}{\partial z} = \frac{\lambda_1 + 1}{2} \left(\frac{\partial u}{\partial x} - \imath \frac{\partial u}{\partial y} \right), \quad z \in D_k. \end{aligned} \quad (2.6)$$

Differentiating (2.5) we arrive at the following problem

$$\psi(t) = \psi_k(t) + \rho \left(\frac{r}{t - a_k} \right)^2 \overline{\psi_k(t)} - 1, \quad |t - a_k| = r, \quad k = 1, 2, ..., N. \quad (2.7)$$

In the above problem we have N contours T_k and N complex conjugation conditions on each contour T_k, and we need to find $N+1$ functions ψ, ψ_1,ψ_2,...,ψ_N, i.e., we need one more condition to close up the system. It is obtained by applying Liouville's theorem for doubly periodic functions.

Let us consider the Banach space C_k of the functions continuous on T_k with the norm $\|\psi_k\| := \max_{T_k} |\psi_k(t)| \ (k = 1, 2, ..., N)$, and the closed subspaces

$C_k^+ \subset C_k$ of functions ψ_k which admit an analytic continuation into D_k. Let us introduce the Banach space C^+ consisting of the functions $\Psi(t) := \psi_k(t) \in C_k^+$ for all $k = 1, 2, ..., N$ with the norm $\|\Psi\| := \max_k \|\psi_k\|$. We use further the same notations for initial functions $\psi_k \in C_k^+$, $\Psi \in C^+$ and their analytic continuation.

By applying a variant of the analytic continuation method (cf., [6]) we arrive at the following system of linear functional equations

$$\psi_m(z) = \rho \sum_{k=1}^{N} \sum_{m_1,m_2}{}^{*} \left(W_{m_1,m_2,k}\psi_k\right)(z) + 1, \qquad |z - a_m| \le r_m, \quad m = 1, 2, ..., N, \tag{2.8}$$

with respect to $\psi_m \in C_m^+$, where

$$\left(W_{m_1,m_2,k}\psi_k\right)(z) = \left(\frac{r}{z - a_k - m_1 - \imath m_1}\right)^2 \overline{\psi_k\left(\frac{r^2}{\overline{t - a_k - m_1 - \imath m_2}} + a_k\right)},$$

and

$$\sum_{k=1}^{N} \sum_{m_1,m_2}{}^{*} W_{m_1,m_2,k} := \sum_{k \ne m} \sum_{m_1,m_2} W_{m_1,m_2,k} + \sum_{m_1,m_2}{}' W_{m_1,m_2,m}. \tag{2.9}$$

Operator $W_{m_1,m_2,k}$ is a compact on C_k^+ for each fixed $m_1, m_2 \in \mathbf{Z}$, $k \in \{1, ..., N\}$. The functional equation (2.8) has a unique solution in C^+. It can be found by the method of successive approximations. The function $\psi(z)$ has then the form

$$\psi(z) = \rho \sum_{k=1}^{N} \sum_{\mathbf{j}} \left(W_{\mathbf{j},k}\psi_k\right)(z), \qquad z \in D_0 \cup \partial D_0. \tag{2.10}$$

Hence an existence and uniqueness of the solution is shown.

In order to give more suitable representation of the unknown functions we look for $\psi_m(z)$ in the form of series in r^2:

$$\psi_m(z) = \psi_m^{(0)}(z) + r^2\psi_m^{(1)}(z) + r^4\psi_m^{(2)}(z) + \dots, \tag{2.11}$$

where each term is expanded into the Taylor series

$$\psi_k^{(s)}(z) = \sum_{l=0}^{\infty} \psi_{lk}^{(s)}(z - a_k)^l.$$

The derivatives $\psi_m^{(p)}(z)$ are defined [2] by following recursive relations:

$$\psi_m^{(0)}(z) = 1,$$

$$\psi_m^{(1)}(z) = \rho \left[\sum_{k \neq m}^{N} \overline{\psi_{0k}^{(0)}} E_2(z - a_k) + \overline{\psi_{0m}^{(0)}} \sigma_2(z - a_m) \right] \tag{2.12}$$

$$= \rho \left[\sum_{k \neq m}^{N} E_2(z - a_k) + \sigma_2(z - a_m) \right],$$

$$\dots$$

$$\psi_m^{(p+1)}(z) = \rho \Big[\sum_{k \neq m}^{N} \overline{\psi_{pk}^{(0)}} E_{p+2}(z - a_k) + \overline{\psi_{pm}^{(0)}} \sigma_{p+2}(z - a_m)$$

$$+ \sum_{k \neq m}^{N} \overline{\psi_{p-1,k}^{(1)}} E_{p+1}(z - a_k) + \overline{\psi_{p-1,m}^{(1)}} \sigma_{p+1}(z - a_m) \tag{2.13}$$

$$+ \dots$$

$$+ \sum_{k \neq m}^{N} \overline{\psi_{0k}^{(p)}} E_2(z - a_k) + \overline{\psi_{0m}^{(p)}} \sigma_2(z - a_m) \Big],$$

where

$$E_m(z) := \sum_{m_1, m_2} (z - m_1 - \imath m_2)^{-m}$$

are so called Eisenstein' functions (the integer numbers m_1, m_2 range from $-\infty$ to $+\infty$ except $m_1^2 + m_2^2 = 0$).

The formulas (2.12), (2.13) describe an algorithm for determination of the current ∇u (u is a solution of the initial problem) in term of the Eisenstein functions, the contrast parameter ρ and the radius r of the disks. Note that to determine the effective conductivity it is sufficient to know the current ∇u.

3. Harmonic Wavelets on the Plane

Our next step is to approximate the Eisenstein' function by using the complex analytic wavelets. A suitable form for the construction of the latter was proposed recently in [8]. We do not go here into the details and describe only the principle step of the construction from [8]. All calculation will be done in the forthcoming papers.

The family of wavelets proposed in [8] has the form

$$\alpha_n(|z|) = 2^{-(j-2)/2} \sum_{\nu}^{\infty} \theta_\varepsilon(\nu 2^{-j-1}) \cos(2\pi\nu(k + 1/2)2^{-j-1}) |z|^\nu \cos \nu x,$$

where $n = 2^j + k$, $0 \leq k < 2^j$, and j is an integer number. Here $\theta_\varepsilon(\omega)$ are "cap-type" functions, i.e. $\theta_\varepsilon(\omega) = 0$ for all $\omega < 1 - \varepsilon, \omega > 1 + \varepsilon$, $\theta_\varepsilon(\omega) = 1$ for all $1 - \varepsilon/2 < \omega < 1 + \varepsilon/2$, and $\theta_\varepsilon(\omega)$ is smooth on the remaining intervals. Such system is used to solve numerically the Dirichlet problem for the Laplace equation in an annulus. In order to apply it for calculation of the Eisenstein' sums we have to make their periodization in two direction (cf., e.g. [4, 9]).

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