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EFFICIENT DOMAIN DECOMPOSITION METHODS AND SHISHKIN-TYPE MESHES FOR SINGULARLY PERTURBED PARABOLIC PROBLEMS ON POLYGONS

L. PORTERO and J.C. JORGE

Departamento Matemática e Informática, Universidad Pública de Navarra
Edificio de las Encinas, Campus de Arrosadía s/n, 31007, Pamplona, Spain
E-mail: laura.portero@unavarra.es;jcjorge@unavarra.es

Abstract. In this work we deal with the numerical resolution of linear evolutionary singularly perturbed problems which are posed on an open convex polygon Ω. In order to obtain a robust and efficient fully discrete algorithm, we first carry out a time integration using a Fractional Step Runge-Kutta method, where the splitting for the elliptic operator is related to a suitable decomposition of Ω. For the spatial discretization, we use certain special meshes which concentrate points in the boundary layers. The keys for the efficiency of the resultant monotone algorithms are that the calculations required for each Fractionary Step can be parallelized and, besides, no Schwarz iteration techniques are required.

Key words: domain decomposition, fractional step Runge-Kutta, Shishkin mesh, singularly perturbed problem, uniform convergence

1. Introduction

Singularly perturbed problems (SPP) arise very often in various areas of sciences and engineering (see [3]). The main peculiarity of the solutions of such problems is the existence of certain narrow zones on the domain (called interior or boundary layers) where they vary very rapidly; the smaller the perturbation parameter ε is, the narrower the layers are and, consequently, the steeper the gradients of the solutions are. It is well known that such behaviour provokes the obtaining of inaccurate numerical solutions if standard discretization techniques are used, unless very fine (ε-dependent) meshes are considered.

The uniform convergence is the key property which a numerical method must satisfy to be suitable for an SPP. This property guarantees acceptable numerical solutions for every value of ε with a computational cost essentially

independent of the size of this parameter. One of the simplest and most succesful techniques for designing uniformly convergent methods was introduced by Shishkin (see [7]). He proposed, firstly for one dimensional problems, certain piecewise uniform meshes which concentrate points in the layers. Then he extended the construction procedure of this type of meshes to rectangles and both standard and advanced discretizations on such special meshes (see [1, 3]) were proven to lead to uniformly convergent schemes, the latter with lower computational cost than the former.

In this framework, an attempt of generalization to problems on polygonal domains was made in [5], where Fractional Step Runge-Kutta (FSRK) schemes, with an operator splitting related to a suitable decomposition of the spatial domain, were combined with finite elements on certain Shishkin-type meshes. This technique led us to new schemes which are easily parallelizable and whose uniform convergence was numerically tested. Nevertheless those algorithms had the drawback of not being monotone due to the existence of obtuse angles in the layer zones of the mesh.

In this paper, we propose to combine similar time discretizations with generalized finite differences on a slightly different Shishkin-type mesh whose angles are all of them less or equal than 90°. This provides numerical algorithms which are not only parallelizable but also monotone and whose uniform convergence has also been numerically tested. Concretely, a numerical test for a diffusion-reaction problem is presented in the last section of the paper.

2. The Problem and the Shishkin-Type Special Meshes

Due to the limitation in size of this paper we shall explain only the following diffusion-reaction case: Find $u : \overline{\Omega} \times [t_0, T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \Delta u + b(\bar{x}, t)\, u = f(\bar{x}, t), \;\; (\bar{x}, t) \in \Omega \times (t_0, T], \\ u(\bar{x}, t_0) = u_0(\bar{x}), \;\; \bar{x} \in \Omega, \\ u(\bar{x}, t) = g(\bar{x}, t), \;\; (\bar{x}, t) \in \Gamma \times (t_0, T], \end{cases} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open convex polygon and $\Gamma = \partial\Omega$. The singular perturbation parameter $0 < \varepsilon << 1$ appears multiplying the diffusion term and $b(\bar{x}, t)$ denotes the reaction coefficient which we assume to be regular enough and to satisfy $b(\bar{x}, t) \geq \beta > 0$. We also suppose that the exact solution of this problem is sufficiently regular both in time and in space and that data f, u_0 and g are smooth enough and satisfy certain compatibility conditions among them.

When ε is much smaller in size than $b(\bar{x}, t)$ parabolic boundary layers $\mathcal{O}(\sqrt{\varepsilon})$ wide appear along all the segments of Γ. Inside each one of these layers the solution varies very quickly in one direction (the perpendicular direction to the boundary segment) and in the intersection of two of them, regions called corner layers, the solution varies very quickly in two directions

(the perpendicular directions to the two boundary segments which intersect in that corner). Regular boundary layers or interior layers are not present under the previous assumptions.

In order to construct a special mesh for Ω, let us consider $N = 4d$, with d a positive integer, and define $\sigma = \min\left\{\frac{1}{4}, \frac{1}{\beta}\sqrt{\varepsilon}\,\log N\right\}$. Firstly we draw in the interior of Ω one parallel segment to each boundary segment at a distance σ. We call them transition segments since they have a similar role to that of the transition points in 1D Shishkin meshes, i.e. they split the fine and the coarse mesh zones.

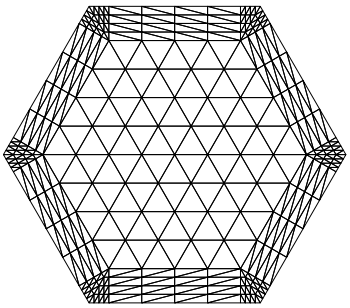

Figure 1. Special mesh for a regular hexagon Ω of side $l = \mathcal{O}(1)$ when $\varepsilon = \mathcal{O}(10^{-2})$, $\beta = 1$ and $N = 16$.

In Fig. 1 we show an example of these special meshes for the case when Ω is a regular hexagon. Note that each one of the three diagonals of Ω shows the structure of a Shishkin 1D mesh, i.e. we consider $\frac{N}{4}$ equidistant points in both ends of the segment (inside the boundary layers) and $\frac{N}{2}$ equidistant points in the interior part of the segment. The coarse mesh inside Ω is constructed as the regular mesh of equilateral triangles which contains the mesh points previously defined on the diagonals. To obtain an appropriate mesh for the quick variation zone, $\frac{N}{4} - 1$ parallel equidistant segments are firstly drawn between each boundary segment and its corresponding transition segment. Then, drawing $\frac{3N}{4} - 1$ perpendicular segments to each boundary side of Ω, as it is shown in Fig. 1, a mesh consisting of triangles and rectangles is obtained. At this point, if we divide each one of such rectangles into two triangles we obtain a triangular mesh which is appropriate to solve our diffusion-reaction problem since it is fine enough in the appropriate directions. Note that this technique can be easily generalized to other convex polygonal domains.

3. The Totally Discrete Algorithm

The IBVP (2.1) admits the following operational formulation: Find $u : [t_0, T] \rightarrow \mathcal{H}$ such that

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)\, u(t) + f(t),\ t \in (t_0, T], \\ u(t_0) = u_0 \in \mathcal{H}, \quad Bu(t) = g(t) \in \mathcal{H}^b, \end{cases}$$

where $\mathcal{H}$ and $\mathcal{H}^b$ are spaces of functions defined on Ω and Γ, respectively. $A(t) : \mathcal{D} \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is the elliptic differential operator $A(t) = \varepsilon \Delta - b(\bar{x}, t)\, \mathcal{I}$ and $B : \mathcal{D} \subseteq \mathcal{H} \longrightarrow \mathcal{H}^b$ is an abstract trace operator.

In order to use an FSRK method as time integrator we should consider certain decompositions for A and f:

$$A(t) = \sum_{i=1}^{m} A_i(t), \quad f(t) = \sum_{i=1}^{m} f_i(t).$$

Then, for τ (constant) time step, approximations $u_n(\bar{x}) \approx u(\bar{x}, t_n)$ are obtained by solving:

$$\begin{cases} \begin{cases} U_n^j = u_n + \tau \displaystyle\sum_{k=1}^{j} a_{jk}^{i_k} \left(A_{i_k}(t_{n,k}) U_n^k + f_{i_k}(t_{n,k}) \right), \\ B_{i_j} U_n^j = g(t_{n,j}), \qquad \text{for } j = 1, \ldots, s, \end{cases} \\ u_{n+1} = u_n + \tau \displaystyle\sum_{j=1}^{s} b_j^{i_j} \left(A_{i_j}(t_{n,j}) U_n^j + f_{i_j}(t_{n,j}) \right),\ n = 0, 1, \ldots, n_f, \end{cases} \tag{3.1}$$

where $i_j \in \{1, \ldots, m\}$, $t_n = t_0 + n\tau$, $t_{n,j} = t_n + c_j \tau$, $n_f = \left[\dfrac{T}{\tau} \right] - 1$. The improvement which these methods provide with respect to standard time integrators such as Runge-Kutta schemes comes from the fact that the operator which acts implicitly in the *j-th* internal stage is $A_{i_j}(t)$ instead of $A(t)$.

The splitting chosen in this work is related to a decomposition of the spatial domain Ω as the union of certain overlapped subdomains $\Omega = \bigcup_{i=1}^{m} \Omega_i$. Each one of these subdomains consists of the union of a certain number of disjoint components $\Omega_i = \bigcup_{j=1}^{m_i} \Omega_{ij}$, with $\Omega_{ij} \cap \Omega_{ik} = \emptyset$ if $j \neq k$ (see [4]). Let us construct a sufficiently smooth partition of unity $\{\psi_i(\bar{x})\}_{i=1}^{m}$ such that $\psi_i(\bar{x})$ takes the value 0 outside subdomain Ω_i, varies smoothly between 0 and 1 in the overlaps of Ω_i with the rest of the subdomains and takes the value 1 in the points which belong just to subdomain Ω_i and not to any other subdomain. From this partition of unity we can define

$$\begin{aligned} A_i(\bar{x}, t) &= \varepsilon\, div\, (\psi_i(\bar{x})\ \nabla)\ - \psi_i(\bar{x})\, b(\bar{x}, t)\, \mathcal{I}, \\ f_i(\bar{x}, t) &= \psi_i(\bar{x}) f(\bar{x}, t)\ \ \forall\, i = 1, \ldots, m. \end{aligned}$$

Finally, for $j \in \{1,\dots,s\}$ and $i_j \in \{1,\dots,m\}$, $B_{i_j}U_n^j$ means to evaluate U_n^j on $\overline{\Omega}_{i_j} \cap \Gamma$.

Hence, when combining the previous time integration with a suitable spatial discretization, the calculation of the *j-th* stage consists of the resolution of a linear system with a number of unknowns equal to the number of mesh points in subdomain Ω_{i_j} and not to the total number of mesh points, as it happens when using a classical time integrator. Moreover, as Ω_{i_j} consists of the union of m_{i_j} disjoint connected components, such system will be actually a collection of m_{i_j} smaller uncoupled linear systems which can be solved in parallel. In the mesh points outside of Ω_{i_j} we just have to update the values of the unknowns according to $U_n^j = F_n^j$ where

$$F_n^j \equiv u_n + \tau \sum_{k=1}^{j-1} a_{jk}^{i_k} \left(A_{i_k}(t_{n,k}) U_n^k + f_{i_k}(t_{n,k}) \right) + a_{jj}^{i_j} f_{i_j}(t_{n,j}).$$

There is also an improvement in using this technique with respect to the use of domain decomposition methods for parabolic problems (see [6]) which is that it does not require any iterative process.

The time integrator used in the numerical examples included in the following section is the Fractionary Implicit Euler scheme. It is first order consistent and stable even for a family of non-commutative operators $\{A_i(t)\}_{i=1}^m$. This method provides numerical approximations $u_{n+1} \approx u(t_{n+1})$ by solving

$$\begin{cases} \begin{cases} (\mathcal{I} - \tau A_j(t_{n+1}))U_n^j = U_n^{j-1} + \tau f_j(t_{n+1}), \quad (U_n^0 = u_n), \\ B_j U_n^j = g(t_{n+1}), \qquad \text{for } j = 1,\dots,m, \end{cases} \\ u_{n+1} = U_n^m, \quad n = 0,1,\dots,n_f. \end{cases}$$

Note that this method is included into the family of FSRK schemes since it can be written as (3.1) if we consider $s = m$, $i_k = k \ \forall k \in \{1,\dots,m\}$ and $a_{jk}^{i_k} = b_j^{i_j} = c_j = 1 \ \forall j,k \in \{1,\dots,m\}$.

Hence, the resolution of each internal stage consists of solving the elliptic equation

$$(\mathcal{I} - \tau\varepsilon\, div(\psi_j \nabla) + \tau\psi_j \mathcal{I})U_n^j = U_n^{j-1} + \tau\psi_j f(t_{n+1})$$

together with the boundary condition $U_n^j|_{\overline{\Omega}_j \cap \Gamma} = g(t_{n+1})|_{\overline{\Omega}_j \cap \Gamma}$. The spatial discretization of such boundary value problems will be carried out by using generalized finite differences (see [2]). Let $\mathcal{T}_N$ be the triangulation of Ω defined by the special mesh introduced in the previous section and let P_0 be an interior node of $\mathcal{T}_N$ with N_0 neighbour nodes $(P_1,\dots,P_{N_0})$. Given $i \in \{1,\dots,N_0\}$, $K_{0,i,i+1}$ denotes the triangle of vertices P_0, P_i, P_{i+1} (considering $P_{N_0+1} \equiv P_1$) and $B_{0,i,i+1}$ denotes the barycenter of such triangle. If we define V_0 as the convex polygon of vertices $\{B_{0,i,i+1}\}_{i=1}^{N_0}$ and $A(V_0)$ denotes the area of V_0, the difference equation for node P_0 is:

$$\tau\varepsilon\psi_j(P_0)\sum_{i=1}^{N_0}\frac{1}{4\,A(K_{0,i,i+1})}\,\frac{|\overline{P_0P_i}|^2-|\overline{P_iP_{i+1}}|^2-|\overline{P_0P_{i+1}}|^2}{2}\,(U^j_{n,i}-U^j_{n,0})$$
$$+\tau\varepsilon\psi_j(P_0)\sum_{i=1}^{N_0}\frac{1}{4\,A(K_{0,i,i+1})}\,\frac{|\overline{P_0P_{i+1}}|^2-|\overline{P_iP_{i+1}}|^2-|\overline{P_0P_i}|^2}{2}\,(U^j_{n,i+1}-U^j_{n,0})$$
$$+(1+\tau\psi_j(P_0))A(V_0)\,U^j_{n,0}\;=\;(U^{j-1}_{n,0}+\tau\psi_j(P_0)f(P_0,t_{n+1}))A(V_0), \tag{3.2}$$

where, given $P,Q\in\mathbb{R}^2$, $|\overline{PQ}|$ denotes the distance between P and Q.

For each $j\in\{1,\dots,m\}$, $U^j_{n,k}\approx U^j_n(P_k)$ and finally $u_{n+1,i}=U^m_{n,i}\approx u(P_i,t_{n+1})$. The numerical algorithm corresponding to a general FSRK integration would come from a spatial discretization of type (3.2) applied to each internal stage

$$(\mathcal{I}-\tau\varepsilon a^{i_j}_{jj}\,div(\psi_{i_j}\nabla)+\tau a^{i_j}_{jj}\psi_{i_j}\mathcal{I})U^j_n=F^j_n,$$
$$U^j_n|_{\overline{\Omega}_{i_j}\cap\Gamma}=g(t_{n+1})|_{\overline{\Omega}_{i_j}\cap\Gamma}.$$

4. Numerical Examples

Let us now face the resolution of a problem of type (2.1) where $t_0=0$, $T=2$,

$$b(x,y,t)=1+x^2+y^2+\sin\Big(\frac{\pi}{2}t\Big),\,u_0(x,y)=1,\quad g(x,y,t)=1,$$

$$f(x,y,t)=3te^{-3t+1}\cos\Big(\frac{\pi y}{\sqrt{3}}\Big)\cos\Big(\frac{\pi\sqrt{3}}{6}\Big(y+\sqrt{3}x\Big)\Big)\cos\Big(\frac{\pi\sqrt{3}}{6}\Big(y-\sqrt{3}x\Big)\Big).$$

The domain Ω is the regular hexagon of vertices $(-\frac{1}{2},\frac{\sqrt{3}}{2})$, $(\frac{1}{2},\frac{\sqrt{3}}{2})$, $(1,0)$, $(\frac{1}{2},-\frac{\sqrt{3}}{2})$, $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$, $(-1,0)$. As we are interested in checking the uniformly convergent behaviour of our scheme and not in showing the reduction in computational cost derived from parallelization, we consider a very simple domain decomposition $\Omega=\bigcup_{i=1}^6\Omega_i$. Concretely, $\{\Omega_i\}_{i=1}^6$ are six equilateral triangles (whose three vertices are the two vertices of one side of the hexagon and its center) extended with a band, which is $d=\frac{1}{8}$ in width, added along the sides of these triangles which are inside Ω. These bands are the overlaps between the subdomains $\{\Omega_i\}_{i=1}^6$. From the univariate function

$$i(w)=\begin{cases}0, & \text{if } -\frac{\sqrt{3}}{2}\le w\le-\frac{\sqrt{3}d}{2},\\ \frac{1}{2}+\frac{\sqrt{3}}{2d}w-\frac{2\sqrt{3}}{9d^3}w^3, & \text{if } -\frac{\sqrt{3}d}{2}<w<\frac{\sqrt{3}d}{2},\\ 1, & \text{if } \frac{\sqrt{3}d}{2}\le w\le\frac{\sqrt{3}}{2},\end{cases}$$

we construct the following partition of unity:

$$\psi_1(x,y)=i(w_1)\,i(w_2)\,(1-i(w_3)),\quad \psi_2(x,y)=i(w_1)\,i(w_2)\,i(w_3),$$
$$\psi_3(x,y)=(1-i(w_1))\,i(w_2),\quad \psi_4(x,y)=(1-i(w_1))\,(1-i(w_2))\,i(w_3),$$
$$\psi_5(x,y)=(1-i(w_1))\,(1-i(w_2))\,(1-i(w_3)),\quad \psi_6(x,y)=i(w_1)\,(1-i(w_2)),$$

Table 1. Maximum global errors and numerical orders of convergence ($N\tau = 0.2$).

	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
$\varepsilon = 1$	7.076E-3	8.143E-3	4.770E-3	2.669E-3	1.412E-3	7.233E-4
	-0.2027	0.7716	0.8375	0.9191	0.9651	
$\varepsilon = 10^{-2}$	3.485E-2	2.275E-2	9.712E-3	3.980E-3	1.169E-3	3.087E-4
	0.6154	1.2278	1.2871	1.7674	1.9209	
$\varepsilon = 10^{-4}$	3.420E-2	2.247E-2	9.539E-3	3.925E-3	1.422E-3	4.837E-4
	0.6060	1.2361	1.2814	1.4651	1.5552	
$\varepsilon = 10^{-6}$	3.424E-2	2.247E-2	9.537E-3	3.921E-3	1.420E-3	4.832E-4
	0.6076	1.2363	1.2853	1.4652	1.5554	
$\varepsilon = 10^{-8}$	3.424E-2	2.247E-2	9.538E-3	3.920E-3	1.420E-3	4.832E-4
	0.6078	1.2363	1.2827	1.4652	1.5554	

where $w_1 = y,\ w_2 = \frac{1}{2}(y + \sqrt{3}x)$, and $w_3 = -\frac{1}{2}(y - \sqrt{3}x)$.

With the previous data, the solution of the problem shows parabolic boundary layers along the whole boundary Γ. The estimations of the maximum global errors have been computed using the double mesh principle in time and in space, i. e., we define $E_{\tau,N} \equiv \max_{n,i} \left| u_{n,i}^{\tau,N} - u_{n,i}^{\frac{\tau}{2},2N} \right|$ where $u_{n,i}^{\tau,N} \approx u(P_i, t_n)$ has been obtained with a discretization of type (3.2) after n steps of size τ on a mesh with parameter N and $u_{n,i}^{\frac{\tau}{2},2N} \approx u(P_i, t_n)$ has been obtained with a discretization of type (3.2) after $2n$ steps of size $\frac{\tau}{2}$ on a mesh with parameter $2N$ (the value of σ for the fine mesh has been calculated using N and not $2N$ in order to obtain a fine mesh which contains the thick one; this avoids the need to interpolate). The numerical orders of convergence rate have been computed as $p_{\tau,N} = \log_2 \frac{E_{\frac{\tau}{2},2N}}{E_{\tau,N}}$. In Tab. 1 we show estimations of the global errors as well as their corresponding numerical orders of convergence for different values of N and ε (τ chosen such that $N\tau = 0.2$).

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