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FINITE-DIFFERENCE SCHEMES OR FINITE ELEMENT METHOD FOR WEAKLY COMPRESSIBLE GAS

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Abstract. An implicit finite difference scheme with a splitting operator is constructed for a linear system of equations describing an unsteady viscous weakly compressible gas flow in the case of two spatial variables. For the numerical solution of this scheme, an error estimate is obtained depending on the parameter characteristics and gas viscosity. The numerical results presented show the efficiency of this method in comparison with an implicit scheme based on iterative conjugate gradient methods. A finite element scheme is considered and error estimate in this case is obtained.
Key words: viscous weakly compressible gas, economical finite difference scheme.

1. Initial-Boundary Value Problem

Consider the system of linear equations describing an unsteady flow of a viscous weakly compressible barotropic gas:

$$\begin{cases} \dfrac{\partial p}{\partial t} + k\,\mathrm{div}\boldsymbol{u} = 0, \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{f}. \end{cases} \tag{1.1}$$

The sought pressure p and vector $\boldsymbol{u} = (u^1, u^2)$ are functions of Euler variables $(t, x) \in Q_T = [0, T] \times \Omega$. Here, μ denotes the dynamic viscosity, which is considered to be a known positive constant. The function $\boldsymbol{f}$ (the vector of external forces) appearing in the equations is a known function of Euler variables. The parameter k denotes a positive constant characterizing gas compressibility (see [1, 2]). We consider the gas to be weakly compressible if $k > 1$.

System (1.1) is supplemented by the initial and boundary conditions

$$
\begin{aligned}
&(p, \boldsymbol{u})|_{t=0} = (p_0, \boldsymbol{u}_0),\ x \in \Omega, \\
&\boldsymbol{u}(t,x) = \mathbf{0},\ (t,x) \in [0,T] \times \partial\Omega.
\end{aligned} \tag{1.2}
$$

Below, we assume that there exists a unique solution to problem (1.1), (1.2).

2. Notation and Auxiliary Statements

Let $\Omega = [0, l_1] \times [0, l_2]$, $h_1 = l_1/M_1$, $h_2 = l_2/M_2$ and

$$
\begin{aligned}
&\overline{\Omega}_1^h = \{(ih_1, jh_2) : 0 \le i \le M_1, 0 \le j \le M_2\}, \\
&\Omega_2^h = \{((i+1/2)h_1, (j+1/2)h_2) : 0 \le i \le M_1 - 1, 0 \le j \le M_2 - 1\}.
\end{aligned}
$$

Let Ω_1^h and $\partial\Omega_1^h$ denote the sets of interior and boundary points of $\overline{\Omega}_1^h$, respectively. Assume that $\overline{\mathrm{U}}_1^h$ is space of functions $U_{i,j}$, defined on $\overline{\Omega}_1^h$ and vanishing on $\partial\Omega_1^h$, U_1^h is space of functions $U_{i,j}$, defined on Ω_1^h, and P_2^h is the space of functions $P_{i+\frac12,j+\frac12}$, defined on Ω_2^h, and $\overline{\mathbf{U}}_1^h = \overline{\mathrm{U}}_1^h \times \overline{\mathrm{U}}_1^h$, $\mathbf{U}_1^h = \mathrm{U}_1^h \times \mathrm{U}_1^h$ are the linear space of vector functions.

Then, define the difference differentiation operators $\widetilde{\partial}_x^{12}$, $\widetilde{\partial}_y^{12}$: $\overline{\mathrm{U}}_1^h \to \mathrm{P}_2^h$:

$$
\begin{aligned}
&(\widetilde{\partial}_x^{12} U)_{i+\frac12,j+\frac12} = \frac12\Big(\frac{U_{i+1,j} - U_{i,j}}{h_1} + \frac{U_{i+1,j+1} - U_{i,j+1}}{h_1}\Big), \\
&(\widetilde{\partial}_y^{12} U)_{i+\frac12,j+\frac12} = \frac12\Big(\frac{U_{i,j+1} - U_{i,j}}{h_2} + \frac{U_{i+1,j+1} - U_{i+1,j}}{h_2}\Big), \\
&\qquad i = 0, \ldots, M_1 - 1, \quad j = 0, \ldots, M_2 - 1;
\end{aligned}
$$

and $\widetilde{\partial}_x^{21}$, $\widetilde{\partial}_y^{21}$: $\mathrm{P}_2^h \to \mathrm{U}_1^h$:

$$
\begin{aligned}
&(\widetilde{\partial}_x^{21} P)_{i,j} = \frac12\left(\frac{P_{i+\frac12,j+\frac12} - P_{i-\frac12,j+\frac12}}{h_1} + \frac{P_{i+\frac12,j-\frac12} - P_{i-\frac12,j-\frac12}}{h_1}\right), \\
&(\widetilde{\partial}_y^{21} P)_{i,j} = \frac12\left(\frac{P_{i+\frac12,j+\frac12} - P_{i+\frac12,j-\frac12}}{h_2} + \frac{P_{i-\frac12,j+\frac12} - P_{i-\frac12,j-\frac12}}{h_2}\right), \\
&\qquad i = 1, \ldots, M_1 - 1, \quad j = 1, \ldots, M_2 - 1;
\end{aligned}
$$

and the difference divergence $\mathrm{div}^h, \widetilde{\mathrm{div}}^h : \overline{\mathbf{U}}_1^h \to \mathrm{P}_2^h$, gradient $\nabla^h, \widetilde{\nabla}^h : \mathrm{P}_2^h \to \mathbf{U}_1^h$, and the discrete Laplacian $\Delta^h : \overline{\mathrm{U}}_1^h \to \mathrm{U}_1^h$ operators:

$$
\begin{aligned}
&(\mathrm{div}^h \boldsymbol{U})_{i+\frac12,j+\frac12} = \frac{U^1_{i+1,j} - U^1_{i,j}}{h_1} + \frac{U^2_{i,j+1} - U^2_{i,j}}{h_2}, \\
&\qquad i = 0, \ldots, M_1 - 1, \quad j = 0, \ldots, M_2 - 1;
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^h P)_{i,j} = \Big\{\frac{P_{i+\frac12,j+\frac12} - P_{i-\frac12,j+\frac12}}{h_1}, \frac{P_{i+\frac12,j+\frac12} - P_{i+\frac12,j-\frac12}}{h_2}\Big\}, \\
&\qquad i = 1, \ldots, M_1 - 1, \quad j = 1, \ldots, M_2 - 1;
\end{aligned}
$$

$$\widetilde{\operatorname{div}}^h \boldsymbol{U} = \widetilde{\partial}_x^{12} U^1 + \widetilde{\partial}_y^{12} U^2, \quad \widetilde{\nabla}^h P = \left(\widetilde{\partial}_x^{21} P, \widetilde{\partial}_y^{21} P\right), \quad \Delta^h \boldsymbol{U} = (\Delta^h U^1, \Delta^h U^2);$$
$$\Delta^h U = \left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_1^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h_2^2} \right).$$

We use the following norms in spaces of functions $\overline{\mathbf{U}}_1^h$ and P_2^h:

$$\|\boldsymbol{U}\| = \sqrt{h_1 h_2 \sum_{i,j \in \Omega_1^h} (U_{i,j}^1)^2 + (U_{i,j}^2)^2}, \quad \|P\| = \sqrt{h_1 h_2 \sum_{i,j \in \Omega_2^h} (P_{i-1/2,j-1/2})^2}.$$

For functions in $\overline{\mathbf{U}}_1^h$, we will also use the norm $\|\cdot\|_{-1}$ defined in the dual space by the following equality

$$\|U\|_{-1} = \sup_{\varphi \in \overline{\mathbf{U}}_1^h} \frac{|(\varphi, U)|}{|\varphi|_1},$$

where a seminorm $|U|_1$ is the expression $(-\Delta^h U, U)^{0.5}$.

3. Finite-Difference Scheme

In [1, 2], problem (1.1), (1.2) was solved by applying the difference scheme

$$\begin{cases} P_t + k\,\widetilde{\operatorname{div}}^h \hat{\boldsymbol{U}} = 0, \\ \boldsymbol{U}_t + \widetilde{\nabla}^h \hat{P} = \mu \Delta^h \hat{\boldsymbol{U}} + \hat{\boldsymbol{F}}, \end{cases} \tag{3.1}$$

where $\boldsymbol{U} = \boldsymbol{U}^n$, $n = 0, 1, \ldots, N$ $(N\tau = T)$, $\hat{\boldsymbol{U}}^n = \boldsymbol{U}^{n+1}$, $\hat{P}^n = P^{n+1}$,

$$\boldsymbol{U}_t = \frac{\hat{\boldsymbol{U}} - \boldsymbol{U}}{\tau}, \quad P_t = \frac{\hat{P} - P}{\tau},$$

and $\bar{\omega}^\tau = \{l\tau : 0 \le l \le N\}$.

The initial and boundary conditions for scheme (3.1) are defined by the equalities

$$(P, \boldsymbol{U})|_{n=0} = (p_0, \boldsymbol{u}_0), \quad \boldsymbol{U}|_{\partial\Omega_1^h \times \omega^\tau} = \mathbf{0}. \tag{3.2}$$

The Uzawa method is considered a classical algorithm for this purpose. It can be described as follows. The value $\boldsymbol{U}$ is expressed from the second equation in (3.1) and substituted into the first equation to obtain

$$(E - \tau^2 k\,\widetilde{\operatorname{div}}^h (E - \tau\mu\Delta^h)^{-1} \widetilde{\nabla}^h) \hat{P} = -\tau \widetilde{\operatorname{div}}^h (E - \tau\mu\Delta^h)^{-1} (\tau \hat{\boldsymbol{F}} + \boldsymbol{U}) + P, \tag{3.3}$$

$$\hat{\boldsymbol{U}} = -\tau(E - \tau\mu\Delta^h)^{-1} \widetilde{\nabla}^h \hat{P} + (E - \tau\mu\Delta^h)^{-1} (\tau \hat{\boldsymbol{F}} + \boldsymbol{U}). \tag{3.4}$$

Thus, the problem is reduced to equations (3.3) and (3.4), which are solved sequentially. Let define

$$G = -\tau \widetilde{\operatorname{div}}^h (E - \tau\mu\Delta^h)^{-1} (\tau \hat{\boldsymbol{F}} + \boldsymbol{U}) + P,$$
$$B_0^h = (E - \tau^2 k\,\widetilde{\operatorname{div}}^h (E - \tau\mu\Delta^h)^{-1} \widetilde{\nabla}^h).$$

The equations written show that the most laborious part in finding a difference solution on the upper time level is solving the equation $B_0^h \hat{P} = G$. In this paper, we propose a modification of scheme (3.1) based on replacing B_0^h with a splitting operator:

$$\big(E - \tau^2 k\, \widetilde{\partial}_x^{12}(E - \tau\mu\Delta^h)^{-1}\widetilde{\partial}_x^{21}\big)\big(E - \tau^2 k\, \widetilde{\partial}_y^{12}(E - \tau\mu\Delta^h)^{-1}\widetilde{\partial}_y^{21}\big)\hat{P} = G. \quad (3.5)$$

Then, the difference scheme takes the form

$$\begin{cases} P_t + k\,\widetilde{\operatorname{div}}^h \hat{\boldsymbol{U}} + k^2\tau^3 B^h \hat{P} = 0, \\ \boldsymbol{U}_t + \widetilde{\nabla}^h \hat{P} = \mu\Delta^h \hat{\boldsymbol{U}} + \hat{\boldsymbol{F}}, \end{cases} \quad (3.6)$$

where

$$B^h q = \widetilde{\partial}_x^{12}(E - \tau\mu\Delta^h)^{-1}\widetilde{\partial}_x^{21}\widetilde{\partial}_y^{12}(E - \tau\mu\Delta^h)^{-1}\widetilde{\partial}_y^{21} P.$$

As before, the initial and boundary conditions are given by (3.2). This difference scheme was considered in [3].

Theorem 1. *A solution to difference scheme* (3.6), (3.2) *exists and is unique.*

4. Error Analysis of the Finite-Difference Scheme

Let ψ_1 and ψ_2 denote the residuals arising when the mesh projections of the exact solution to problem (1.1), (1.2) are substituted into equations (3.6):

$$\psi_1 = \frac{1}{k}\left(\frac{\partial p}{\partial t} - p_{\bar{t}}\right) + \operatorname{div}\boldsymbol{u} - \widetilde{\operatorname{div}}^h \boldsymbol{u} - k\tau^3 B^h p, \quad \check{p}^n = p^{n-1}, p_{\bar{t}} = \frac{p - \check{p}}{\tau}$$

$$\boldsymbol{\psi}_2 = \mu(\Delta^h \boldsymbol{u} - \Delta\boldsymbol{u}) + \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u}_{\bar{t}} + \nabla p - \widetilde{\nabla}^h p, \quad \check{\boldsymbol{u}}^n = \boldsymbol{u}^{n-1}, \boldsymbol{u}_{\bar{t}} = \frac{\boldsymbol{u} - \check{\boldsymbol{u}}}{\tau}.$$

Theorem 2. *The difference between the difference solution and the exact solution to problem satisfies the estimate*

$$R_1 \equiv \max_{n=1}^{N}\left(\frac{1}{\sqrt{k}}\|p^n - P^n\| + \|\boldsymbol{u}^n - \boldsymbol{U}^n\| + \sqrt{\tau\mu\sum_{m=1}^{n}\|\nabla^h(\boldsymbol{u}^m - \boldsymbol{U}^m)\|^2}\right)$$
$$\le \sqrt{3}\left(\frac{1}{\sqrt{k}}\|p^0 - P^0\| + \|\boldsymbol{u}^0 - \boldsymbol{U}^0\| + 2\sqrt{2}\tau\sum_{n=1}^{N}\left(\sqrt{k}\|\psi_1^n\| + \|\boldsymbol{\psi}_2^n\|\right)\right).$$

Corollary 1. The smooth solution $(\boldsymbol{u}, p)$ to the differential problem satisfies the estimate

$$R_1 \le C\,\mathrm{e}^T\left(\tau + (1 + \sqrt{k})h^2 + \left(\sqrt{k\tau/\mu}\right)^3\right).$$

In fact, Theorem 2 fails to estimate the error in p for large k. A more precise estimate of this error is given by the following theorem.

Theorem 3. *The difference between the difference solution and the exact solution to problem satisfies the estimate*

$$
\begin{aligned}
R_2 \equiv \max_{n=1}^{N}\Big(\|p^n-P^n\|+\|\boldsymbol{u}^n-\boldsymbol{U}^n\|+\frac{\|p^n_{\bar t}-P^n_{\bar t}\|}{\sqrt{k}\mu}+\frac{\|\boldsymbol{u}^n_{\bar t}-\boldsymbol{U}^n_{\bar t}\|}{\mu}+|\boldsymbol{u}^n-\boldsymbol{U}^n|_1\Big)\\
\le C\Big(\frac{(\mu+\sqrt{\mu T}+\sqrt{k})\tau+1}{\tau\mu}\Big(\|p^0-P^0\|+|(p^0-P^0,1)|+\|\boldsymbol{u}^0-\boldsymbol{U}^0\|\\
+\sqrt{\tau\mu}|\boldsymbol{u}^0-\boldsymbol{U}^0|_1\Big)+\frac{1+\sqrt{T\mu}}{\mu}h^2+\sqrt{\mu\tau\sum_{n=1}^{N}\|\psi_1^n\|^2}+\sqrt{\frac{\tau}{\mu}\sum_{n=1}^{N}\|\boldsymbol{\psi}_2^n\|_{-1}^2}\\
+\sqrt{\frac{\tau}{\mu^3}\sum_{n=2}^{N-1}\|(\psi_1^n)_{t\bar t}\|^2}+\sqrt{\frac{\tau}{\mu^3}\sum_{n=2}^{N}\|(\boldsymbol{\psi}_2^n)_{\bar t}\|_{-1}^2}+\sqrt{\frac{\tau}{\mu}\sum_{n=1}^{N}\|\boldsymbol{\psi}_2^n\|^2}\\
+\frac{1}{\mu}\left(\max_{n=1}^{N}(\|\boldsymbol{\psi}_2^n\|_{-1}+\|(\psi_1^n)_{\bar t}\|)+(1+\tau\sqrt{k})(\sqrt{k}\|\psi_1^1\|+\|\boldsymbol{\psi}_2^1\|)\right)\Big).
\end{aligned}
$$

Corollary 2. For a smooth solution $(\boldsymbol{u}, p)$ of the differential problem the error can be estimated as

$$
\begin{aligned}
R_2 \le C\Big(\frac{1+\sqrt{T\mu}}{\mu}h^2+\Big(\sqrt{\mu}+\frac{1}{\sqrt{\mu^3}}+\frac{\tau\sqrt{k}}{\mu}\Big)+\tau\\
+h^2+k\sqrt{\tau^3/\mu^3}+\frac{(1+\tau\sqrt{k})\sqrt{k}}{\mu}(h^2+k\sqrt{\tau^3/\mu^3})\Big).
\end{aligned}
$$

Remark 1. If $\tau\sqrt{k}$ is bounded, then inequality takes the form

$$
R_2 \le C\left(\tau+(1+\sqrt{k})\left(h^2+k\tau^{3/2}\right)\right),
$$

where the dependence of the differential solution on μ and k is not shown explicitly but is hidden in C.

5. Numerical Results

To illustrate the theoretical results, problem (1.1), (1.2) in $\Omega=[0,1]\times[0,1]$ was solved numerically for various k and μ. Consider the case, when problem (1.1), (1.2) has the smooth solution

$$
\begin{aligned}
p &= -\cos(\tfrac{k\pi}{t+2.5})(\cos(\pi x)\sin(2\pi y)+\sin(2\pi x)\cos(\pi y)),\\
u^1 &= \tfrac{1}{t+2.5}\sin(\tfrac{k\pi}{t+2.5})\sin(\pi x)\sin(2\pi y),\\
u^2 &= \tfrac{1}{t+2.5}\sin(\tfrac{k\pi}{t+2.5})\sin(2\pi x)\sin(\pi y).
\end{aligned}
$$

We implemented schemes (3.1) and (3.6). A solution to the former was computed using the conjugate gradient method. The numerical results have shown that the scheme with a splitting operator is especially efficient in comparison with the implicit scheme described when k is large. The numerical

Table 1.

	SM	CG	SM	CG	SM	SG
τ/h	1/32	1/32	1/64	1/64	1/128	1/128
1/32	3.8e-1	—	3.8e-1	—	3.8e-1	—
1/64	9.9e-2	—	9.4e-2	—	9.4e-2	—
1/128	1.7e-2	5.8e-3	1.6e-2	—	1.6e-2	—
1/256	1.3e-2	7.5e-4	1.0e-2	9.2e-3	1.0e-2	4.1e-3

Table 2.

	SM	CG	SM	CG	SM	SG
τ/h	1/32	1/32	1/64	1/64	1/128	1/128
1/32	6.2e-1	—	6.4e-1	—	6.4e-1	—
1/64	7.8e-2	—	8.7e-2	—	8.9e-2	—
1/128	1.5e-2	3.8e-1	1.6e-2	—	1.6e-2	—
1/256	5.5e-3	3.6e-1	5.5e-3	1.2e-1	6.9e-3	9.0e-2

Table 3.

	SM	CG	SM	CG	SM	SG
τ/h	1/32	1/32	1/64	1/64	1/128	1/128
1/32	2.8e-3	1.0e-2	2.6e-3	1.2e-2	3.0e-3	1.3e-2
1/64	2.9e-3	3.0e-3	7.0e-4	5.6e-3	7.8e-4	6.3e-3
1/128	3.3e-3	4.3e-4	6.5e-4	2.3e-3	2.0e-4	2.9e-3
1/256	3.4e-3	1.9e-3	7.6e-4	6.8e-4	1.1e-4	1.3e-3

results for $k = 30$ and $\mu = 10^{-2}$ are presented in Tab. 1 and Tab. 2. They contain the L_2 norms of the errors in pressure and the first component of velocity, respectively. A dash in both tables means that no solution to difference scheme (3.1) was found. With these values of k and μ, the conjugate gradient method does not converge for large time steps, while the splitting method, which is not. The divergence of the conjugate gradient method is apparently connected with the fact that the matrix of the system is ill conditioned and there exist roundoff errors in computer calculations.

At the same time, for small k, the splitting method is not inferior to the implicit scheme. This can be seen from Tab. 3 and Tab. 4, which list the L_2 norms of the errors in pressure and the first component of velocity, respectively, for $k = 1$ and $\mu = 10^{-1}$. It follows from the aforesaid that, for large k, scheme (3.6) with a splitting operator is preferable to implicit scheme (3.1) whose solution is sought by the conjugate gradient method, while, for small k, the numerical results for both schemes are much the same.

Table 4.

	SM	CG	SM	CG	SM	SG
τ/h	1/32	1/32	1/64	1/64	1/128	1/128
1/32	4.0e-2	6.6e-2	3.2e-2	5.5e-2	2.9e-2	5.2e-2
1/64	2.3e-2	3.9e-2	1.3e-2	2.8e-2	1.0e-2	2.6e-2
1/128	1.8e-2	2.5e-2	7.7e-3	1.5e-2	5.2e-3	1.2e-2
1/256	1.5e-2	1.9e-2	5.5e-3	9.2e-3	2.9e-3	6.7e-3

6. Finite Element Method

Since finite difference schemes make stringent demands to the smoothness of the problem (1.1)-(1.2), we will consider application of the finite elements method which makes lower demands to smoothness. We will decompose Ω to the several "big" rectangles, then each rectangle we decompose to 4 equal "small" rectangles. We define piecewise constant function spaces P_h for "big" rectangle and piecewise linear function space $\mathbf{U}_h$ for "small" rectangles. Then we obtain the next system of equation

$$\begin{aligned}
&(q_{\bar{t}}, \psi)_\Omega + k(\operatorname{div} \boldsymbol{v}, \psi)_\Omega = 0,\\
&(\boldsymbol{v}_{\bar{t}}, \varphi)_\Omega - (q, \operatorname{div} \varphi)_\Omega + \mu(\nabla \boldsymbol{v}, \nabla \varphi)_\Omega = (\boldsymbol{f}, \varphi)_\Omega,
\end{aligned}$$

where $\psi \in P_h$, $\varphi \in \mathbf{U}_h$, with the initial conditions

$$(q^0, \boldsymbol{v}^0) = (p_h(0), \boldsymbol{u}_h(0)).$$

Boundary conditions for $\boldsymbol{v}$ are satisfied automatically.

The linear system of algebraic equation with variables q, $\boldsymbol{v} = (v^1, v^2)$:

$$\begin{aligned}
&q_{\bar{t}} + k(B^T \boldsymbol{v})^1 + k(B^T \boldsymbol{v})^2 = 0,\\
&Cv^1_{\bar{t}} - B^1 q + \mu A v^1 = f^1,\\
&Cv^2_{\bar{t}} - B^2 q + \mu A v^2 = f^2,
\end{aligned}$$

where f^1_m, f^2_m are the m normed components of the projection $\boldsymbol{f}$ onto the basis $\mathbf{U}_h$,

$$
\begin{aligned}
(Av^l)_{ij} = {} & \frac{1}{h_1h_2}\Big[\frac{4(h_1^2+h_2^2)}{3h_1h_2}v_{i,j}^l - \frac{h_1^2+h_2^2}{6h_1h_2}v_{i+1,j+1}^l - \frac{-h_1^2+2h_2^2}{3h_1h_2}v_{i+1,j}^l \\
& - \frac{h_1^2+h_2^2}{6h_1h_2}v_{i+1,j-1}^l - \frac{2h_1^2-h_2^2}{3h_1h_2}v_{i,j+1}^l - \frac{2h_1^2-h_2^2}{3h_1h_2}v_{i,j-1}^l \\
& - \frac{h_1^2+h_2^2}{6h_1h_2}v_{i-1,j+1}^l - \frac{-h_1^2+2h_2^2}{3h_1h_2}v_{i-1,j}^l - \frac{h_1^2+h_2^2}{6h_1h_2}v_{i-1,j-1}^l\Big], \\
& i = 1,\dots,2M_1-1, \quad j = 1,\dots,2M_2-1, l = 1,2; \\
(B^1q)_{ij} = {} & \frac{1}{2h_1}\begin{cases} -q_{ij}+q_{i-2,j}+q_{i-2,j-2}-q_{i,j-2}, & \text{for } i=2l, j=2m, \\ -2q_{i,j-1}+2q_{i-2,j-1}, & \text{for } i=2l, j=2m+1, \\ 0, & \text{for } i=2l+1, \end{cases} \\
(B^2q)_{ij} = {} & \frac{1}{2h_2}\begin{cases} -q_{ij}-q_{i-2,j}+q_{i-2,j-2}+q_{i,j-2}, & \text{for } i=2l, j=2m, \\ -2q_{i-1,j}+2q_{i-1,j-2}, & \text{for } i=2l+1, j=2m, \\ 0, & \text{for } j=2m+1, \end{cases} \\
& i = 1,\dots,2M_1-1, \quad j = 1,\dots,2M_2-1; \\
(B^T\boldsymbol{v})_{ij} = {} & \frac{v_{i+2,j+2}^1+2v_{i+2,j+1}^1+v_{i+2,j}^1-v_{i,j+2}^1-2v_{i,j+1}^1-v_{ij}^1}{8h_1} \\
& + \frac{v_{i+2,j+2}^2+2v_{i+1,j+2}^2+v_{i,j+2}^2-v_{i+2,j}^2-2v_{i+1,j}^2-v_{ij}^2}{8h_2}, \\
& i = 0,2,\dots,2M_1-2, \quad j = 0,2,\dots,2M_2-2; \\
(Cv)_{ij} = {} & \frac{16v_{ij}+v_{i-1,j+1}+4v_{i,j+1}+v_{i+1,j+1}+4v_{i+1,j}}{36} \\
& + \frac{v_{i+1,j-1}+4v_{i,j-1}+v_{i-1,j-1}+4v_{i-1,j}}{36}.
\end{aligned}
$$

Theorem 4. *A solution of the finite element scheme exists and is unique.*

Theorem 5. *The error the difference solution satisfies the following estimate:*

$$
\begin{aligned}
& \frac{1}{k}\max_{n=1,\dots,N}\|q^n-p^n\| + \max_{n=1,\dots,N}\|\boldsymbol{v}^n-\boldsymbol{u}^n\| + \mu\tau\sum_{n=1}^{N}\|\nabla(\boldsymbol{v}^n-\boldsymbol{u}^n)\| \\
& \le C(\mu,\Omega,T,k)(h+\frac{\sqrt{k}h}{\mu}+\tau).
\end{aligned}
$$

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