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SMOOTHING TRANSFORMATION AND NUMERICAL SOLUTIONS FOR FREDHOLM INTEGRAL EQUATIONS WITH SINGULARITIES[1]

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Abstract. We propose a smoothing technique associated with classical collocation and Galerkin methods for the numerical solution of linear Fredholm integral equations of the second kind with diagonal and boundary singularities.

Key words: Fredholm integral equation, weakly singular kernel, boundary singularity, smoothing transformation

1. Introduction

In many practical applications there arise integral equations of the form

$$u(x) = \int_0^1 K(x,y)u(y)dy + f(x)\,, \quad 0 \le x \le 1\,, \tag{1.1}$$

with $f \in C^m[0,1]$, $K(x,y) = g(x,y)|x-y|^{-\nu}$, $g \in C^m([0,1]\times[0,1])$, $m \in \mathbb{N} = \{1,2,\ldots\}$ and $0 < \nu < 1$ (by $C^k(\Omega)$ we denote the set of all k times continuously differentiable functions on Ω, $C^0(\Omega) = C(\Omega)$). Solutions of this type equations are typically non-smooth at the endpoints of the interval $[0,1]$ where their derivatives become unbounded, see, for example, [1, 6, 7]. In collocation and Galerkin methods the singular behaviour of the exact solution can be taken into account by using special graded grids $\Delta_N^r = \{x_0,\ldots,x_{2N} : 0 = x_0 < \ldots < x_{2N} = 1\}$ with the nodes

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$$x_i = \frac{1}{2}\left(\frac{i}{N}\right)^r, \ i = 0, 1, \ldots, N\,; \quad x_{N+i} = 1 - x_{N-i}, \ i = 1, \ldots, N\,, \qquad (1.2)$$

where $N \in \mathbb{N}$ and $r \in [1, \infty)$. By using a collocation (or Galerkin) method based on piecewise polynomials of degree at most $m - 1$, one can reach a convergence of order $O(N^{-m})$ for sufficiently large values of r, see [1, 6, 7].

However, in practice, the computations on strongly graded grids Δ_N^r for large values of r may be numerically instable since the grid points (1.2) will be located too close one another near the endpoints of $[0, 1]$ and final system of algebraic equations which one has to solve may become rapidly ill-conditioned as N is increased, see [2].

To avoid problems associated with the use of strongly graded grids and still maintain the optimal convergence order, the following approach for the numerical solution of (1.1) can be used. First we regularize the solution to (1.1) by introducing a suitable new independent variable so that the singularities of the derivatives of the solution will be milder or disappear at all. After that we solve the transformed equation by a piecewise collocation (or Galerkin) method on a mildly graded or uniform grid. We refer to [3] for details, compare also [2, 4].

In this contribution we extend the domain of applicability of this technique. To this aim, we examine a more complicated situation for equation (1.1) where the kernel $K(x, y)$, in addition to a diagonal singularity (a singularity as $y \to x$), may have different boundary singularities (singularities as $y \to 0$ and $y \to 1$). Actually, we assume that the kernel K has the form

$$K(x, y) = g(x, y)|x - y|^{-\nu} y^{-\lambda} (1 - y)^{-\mu}, \qquad (1.3)$$

where

$$g \in C^m([0, 1] \times [0, 1])\,, \ m \in \mathbb{N}\,; \ \nu, \lambda, \mu \in (-\infty, 1)\,. \qquad (1.4)$$

The set of kernels satisfying {(1.3),(1.4)} will be denoted by $W^{m,\nu,\lambda,\mu}$. A more general equation with kernels containing diagonal and boundary singularities will be considered in a forthcoming paper.

2. Smoothing Transformation

For given $m \in \mathbb{N}$ and $\alpha, \beta \in (0, 1)$, let $C^{m,\alpha,\beta}(0, 1)$ be the set of all functions $u \in C^m(0, 1)$ such that

$$\left|u^{(j)}(x)\right| \le c_u\left[x^{1-\alpha-j} + (1 - x)^{1-\beta-j}\right], \quad 0 < x < 1, j = 1, \ldots, m\,, \qquad (2.1)$$

where c_u is a positive constant. It follows from (2.1) with $j = 1$ that $u \in C^{m,\alpha,\beta}(0, 1)$ has a continuous extension to $[0, 1]$. The regularity of a solution to (1.1) can be characterized by the following lemma.

Lemma 1. [5] *Let* $K \in W^{m,\nu,\lambda,\mu}$, $f \in C^{m,\nu+\lambda,\nu+\mu}(0, 1)$, $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \le \lambda < 1$, $\nu + \lambda < 1$, $0 \le \mu < 1$, $\nu + \mu < 1$. *Assume also that equation* (1.1) *has a solution* $u \in C[0, 1]$. *Then* $u \in C^{m,\nu+\lambda,\nu+\mu}(0, 1)$.

For $d \in \mathbb{N}$ denote (cf. [3])

$$\varphi(s) = \begin{cases} 2^{d-1}s^d, & \text{if } 0 \le s \le 1/2\,, \\ 1 - 2^{d-1}(1-s)^d, & \text{if } 1/2 \le s \le 1\,. \end{cases} \tag{2.2}$$

Clearly, $\varphi \in C^1[0,1]$ and $\varphi'(s) > 0$ for $0 < s < 1$. Thus, φ maps $[0,1]$ onto $[0,1]$ and has a continuous inverse φ^{-1} with $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(1) = 1$.

In the sequel, c denotes a generic constant.

Lemma 2. *Let* $u_\varphi(s) = u(\varphi(s))$, $s \in [0,1]$, *where* $u \in C^{m,\alpha,\beta}(0,1)$, $m \in \mathbb{N}$, $0 < \alpha < 1$, $0 < \beta < 1$. *Then* $u_\varphi \in C[0,1]$, $u_\varphi\big|_{(0,\frac{1}{2}]} \in C^m(0,\frac{1}{2}]$, $u_\varphi\big|_{[\frac{1}{2},1)} \in C^m[\frac{1}{2},1)$ *and*

$$\left|u_\varphi^{(j)}(s)\right| \le c \begin{cases} 1\,, & \text{if } j \le d(1-\gamma)\,, \\ s^{d(1-\gamma)-j} + (1-s)^{d(1-\gamma)-j}, & \text{if } j > d(1-\gamma)\,, \end{cases} \tag{2.3}$$

where $0 < s < 1$, $\gamma = \max\{\alpha, \beta\}$ *and* $j = 1, \ldots, m$.

Proof. The smoothness claim is clear. Further, for the derivatives of the composite function $u_\varphi = u \circ \varphi$, we have the Faà di Bruno's representation

$$u_\varphi^{(j)}(s) = \sum \frac{j!}{n_1! \ldots n_j!} u^{(n)}(\varphi(s)) \left(\frac{\varphi'(s)}{1!}\right)^{n_1} \cdots \left(\frac{\varphi^{(j)}(s)}{j!}\right)^{n_j}, \tag{2.4}$$

where $n = n_1 + \ldots + n_j$ and the sum is taken over all $n_1, \ldots, n_j \in \{0\} \cup \mathbb{N}$ for which $n_1 + 2n_2 + \ldots + jn_j = j$, $j = 1, \ldots, m$. Since $\varphi^{(j)}(s) \equiv 0$ for $j > d$ it is sufficient to consider only the case $1 \le j \le \min\{d, m\}$.

Let us estimate all terms in (2.4) for $0 < s < 1$. For $0 < s \le 1/2$ we get from (2.1) and (2.2) that

$$\begin{aligned} \left|u^{(n)}(\varphi(s))\right| \varphi'(s)^{n_1} \cdots \varphi^{(j)}(s)^{n_j} &\le c s^{d(1-\alpha-n)+(d-1)n_1+\ldots+(d-j)n_j} \\ &\le c \begin{cases} 1 & \text{for } j \le d(1-\alpha)\,, \\ s^{d(1-\alpha)-j} & \text{for } j > d(1-\alpha)\,. \end{cases} \end{aligned} \tag{2.5}$$

In a similar way we obtain for $1/2 \le s < 1$ that

$$\left|u^{(n)}(\varphi(s))\right| \varphi'(s)^{n_1} \cdots \varphi^{(j)}(s)^{n_j} \le c \begin{cases} 1 & \text{for } j \le d(1-\beta)\,, \\ (1-s)^{d(1-\beta)-j} & \text{for } j > d(1-\beta)\,. \end{cases} \tag{2.6}$$

Estimate (2.3) follows from (2.4)–(2.6). ■

3. Numerical Solution and Convergence Estimates

Using (2.2) we introduce in (1.1) the change of variables $y = \varphi(s)$, $x = \varphi(t)$, $s, t = [0,1]$. We obtain an integral equation in the form

$$u_\varphi(t) = \int_0^1 K_\varphi(t,s) u_\varphi(s) ds + f_\varphi(t)\,, \quad 0 \le t \le 1\,, \tag{3.1}$$

where $f_\varphi(t) = f(\varphi(t))$ and $K_\varphi(t,s) = K(\varphi(t),\varphi(s))\varphi'(s)$ are given functions and $u_\varphi(t) = u(\varphi(t))$ is a function which we have to find.

For given integers $m, N \in \mathbb{N}$ let

$$S_{m-1}^{(-1)}(\Delta_N^r) = \left\{ v : v\big|_{[x_{j-1},x_j]} \in \pi_{m-1}, j = 1,\dots,2N \right\},$$

$$S_{m-1}^{(0)}(\Delta_N^r) = \left\{ v \in C[0,1] : v\big|_{[x_{j-1},x_j]} \in \pi_{m-1}, j = 1,\dots,2N \right\}$$

be two underlying spline spaces of piecewise polynomial functions on the grid Δ_N^r (see (1.2)). Here $v\big|_{[x_{j-1},x_j]}$ $(j = 1,\dots,2N)$ is the restriction of $v(t)$, $t \in [0,1]$, to the subinterval $[x_{j-1},x_j] \subset [0,1]$ and π_{m-1} denotes the set of polynomials of degree not exceeding $m-1$. Note that the elements of $S_{m-1}^{(-1)}(\Delta_N^r)$ may have jump discontinuities at the interior points $x_1,\dots,x_{2N-1}$ of the grid Δ_N^r. In every subinterval $[x_{j-1},x_j]$ $(j = 1,\dots,2N)$ we introduce $m \in \mathbb{N}$ interpolation points

$$x_{jl} = x_{j-1} + \eta_l(x_j - x_{j-1}), \quad l = 1,\dots,m;\ j = 1,\dots,2N, \tag{3.2}$$

where $\eta_1,\dots,\eta_m$ are some fixed parameters which do not depend on j and N and satisfy the conditions $0 \le \eta_1 < \ldots < \eta_m \le 1$.

We find an approximation v to u_φ, the solution of equation (3.1) (under conditions of Theorems 1 and 2 below the equations (1.1) and (3.1) are uniquely solvable), using standard collocation and Galerkin methods. In the first case we find $v = v_{N,m,r,\varphi}$ from the following conditions:

$$v \in S_{m-1}^{(-1)}(\Delta_N^r), \quad N, m \in \mathbb{N},\ r \ge 1, \tag{3.3}$$

$$v(x_{jl}) = \int_0^1 K_\varphi(x_{jl},s)v(s)ds + f_\varphi(x_{jl}), \quad l = 1,\dots,m; j = 1,\dots,2N, \tag{3.4}$$

with x_{jl}, $l = 1,\dots,m; j = 1,\dots,2N$, given by the formula (3.2). In the second case we determine $v = v_{N,m,r,\varphi}$ from the conditions (3.3) and

$$(v - T_\varphi v - f_\varphi, w) = 0 \quad \forall\, w \in S_{m-1}^{(-1)}(\Delta_N^r), \tag{3.5}$$

where T_φ is defined by the formula

$$(T_\varphi z)(t) = \int_0^1 K_\varphi(t,s)z(s)ds, \quad 0 \le t \le 1,$$

and $(\cdot,\cdot)$ denotes the standard inner product for $L^2(0,1)$.

Having determined the approximation v for u_φ, we determine an approximation $u_N = u_{N,m,r,\varphi}$ for u, the solution of equation (1.1), setting

$$u_N(x) = v(\varphi^{-1}(x)), \quad 0 \le x \le 1. \tag{3.6}$$

Remark 1. The choice of nodes (3.2) with $\eta_1 = 0$, $\eta_m = 1$ in (3.4) actually implies that the resulting collocation approximation v belongs to the smoother spline space $S_{m-1}^{(0)}(\Delta_N^r)$ than it is stated by the condition (3.3).

Remark 2. The settings $\{(3.3),(3.4)\}$ and $\{(3.3),(3.5)\}$ form a system of linear algebraic equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Delta_N^r)$. We refer to [3, 4] for a convenient choice of it.

Theorem 1. *Let the following conditions be fulfilled:*
1) $K \in W^{m,\nu,\lambda,\mu}$, $f \in C^{m,\nu+\lambda,\nu+\mu}(0,1)$, $0 < \nu < 1$, $0 \le \lambda < 1$, $\nu + \lambda < 1$, $0 \le \mu < 1$, $\nu + \mu < 1$;
2) equation $u(x) = \int_0^1 K(x,y)u(y)dy$ *has only the trivial solution* $u = 0$;
3) the interpolation nodes (3.2) *with grid points* (1.2) *are used.*

Then equation (1.1) *has a unique solution* $u \in C[0,1]$. *Moreover, for all sufficiently large* $N \in \mathbb{N}$, *say* $N \ge N_0$, *the settings* $\{(3.3),(3.4),(3.6)\}$ *determine a unique approximation* u_N *to* u *and*

$$\sup_{0\le x\le 1} |u_N(x) - u(x)| \le c \begin{cases} N^{-rd(1-\nu-\gamma)}, & \text{if } 1 \le r < \frac{m}{d(1-\nu-\gamma)}, \\ N^{-m}, & \text{if } \quad r \ge \frac{m}{d(1-\nu-\gamma)}, r \ge 1\,, \end{cases} \tag{3.7}$$

where $N \ge N_0$, $\gamma = \max\{\lambda, \mu\}$ *and* c *is a positive constant which is independent of* N.

Proof. We outline the basic ideas on which the proof will be based. The details will be given in a forthcoming paper where a more general situation is considered, compare also [3, 4]. Due to the assumptions 1) and 2), equation (1.1) has a unique solution $u \in C[0,1]$. Further, we write equation (3.1) in the form $u_\varphi = T_\varphi u_\varphi + f_\varphi$ and show that it is uniquely solvable in $L^\infty(0,1)$. It follows from (1.3), (1.4) and (2.2) that $K_\varphi(t,s)$ is continuous for $t,s \in (0,1)$, $t \ne s$ and

$$\left|K_\varphi(t,s)\right| \le c|t-s|^{-\nu} s^{-\lambda}(1-s)^{-\mu}.$$

Since $\nu + \lambda < 1$ and $\nu + \mu < 1$, T_φ is compact as an operator from $L^\infty(0,1)$ into $C[0,1]$, see [5]. Therefore also $u_\varphi \in C[0,1]$. On the basis of Lemmas 1 and 2 we find that for $j = 1,\ldots,m$,

$$\left|u_\varphi^{(j)}(s)\right| \le c \begin{cases} 1\,, & \text{if } j \le d(1-\nu-\gamma)\,, \\ s^{d(1-\nu-\gamma)-j} + (1-s)^{d(1-\nu-\gamma)-j}, & \text{if } j > d(1-\nu-\gamma)\,, \end{cases} \tag{3.8}$$

where $0 < s < 1$ and $\gamma = \max\{\lambda,\mu\}$. Further, conditions (3.3), (3.4) have the operator equation representation

$$v = P_N T_\varphi v + P_N f_\varphi\,, \tag{3.9}$$

where P_N is an operator which assigns to every function $z \in C[0,1]$ its piecewise interpolation function $P_N z \in S_{m-1}^{(-1)}(\Delta_N^r)$ interpolating z at the points (3.2). Using a quite standard arguing (cf. [3, 4, 6]), we obtain that equation (3.9) has a unique solution $v \in S_{m-1}^{(-1)}(\Delta_N^r)$ and

$$\left\|v - u_\varphi\right\|_{L^\infty(0,1)} \le c\left\|P_N u_\varphi - u_\varphi\right\|_{L^\infty(0,1)}, \quad N \ge N_0\,, \tag{3.10}$$

where u_φ is the solution of equation (3.1). Due to (3.8) we get for $N \ge N_0$ that (cf. [6])

$$\left\|P_N u_\varphi - u_\varphi\right\|_{L^\infty(0,1)} \le c \begin{cases} N^{-rd(1-\nu-\gamma)}, & \text{if } 1 \le r < \frac{m}{d(1-\nu-\gamma)}, \\ N^{-m}, & \text{if } \quad r \ge \frac{m}{d(1-\nu-\gamma)}, r \ge 1\,. \end{cases} \tag{3.11}$$

On the basis of (2.2) and (3.6) we have

$$\sup_{0\le x\le 1} \left|u_N(x) - u(x)\right| = \sup_{0\le t\le 1} \left|v(t) - u_\varphi(t)\right|.$$

This together with (3.10) and (3.11) yields the estimate (3.7). ■

Theorem 2. *Let the conditions 1)–2) of Theorem 1 be fulfilled. Then, for all sufficiently large $N \in \mathbb{N}$, say $N \ge N_0$, the settings* {(3.3),(3.5),(3.6)} *determine a unique approximation u_N to u, the solution of equation* (1.1). *Moreover, for $u_N - u$ the estimate* (3.7) *holds.*

We omit the proof since it is completely analogous to that of Theorem 4.2 in [3].

Remark 3. Theorems 1 and 2 suggest that the accuracy $\|u_N - u\|_\infty \le cN^{-m}$ can be achieved on a mildly graded or uniform grid. As an example, if we assume that $\nu = 1/6, \lambda = 1/7, \mu = 1/8, m = 2$ (the case of linear polynomials) and $d = 3$, then the maximal convergence order $\|u_N - u\|_\infty \le cN^{-2}$ is available for $r \ge 1$. In particular, the uniform grid with nodes (1.2), $r = 1$, may be used.

Remark 4. Instead of (2.2) a transformation φ can be introduced which takes into account different singularity orders in (2.3) as $s \to 0$ and $s \to 1$ if $\alpha \ne \beta$. Respectively, the conditions on the parameter r will be different for $[0, 1/2]$ and $[1/2, 1]$.

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