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STURM-LIOUVILLE PROBLEM FOR STATIONARY DIFFERENTIAL OPERATOR WITH NONLOCAL INTEGRAL BOUNDARY CONDITION

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Abstract. The Sturm-Liouville problem with one classical boundary condition and another nonlocal integral boundary condition is considered in this paper. Two cases of nonlocal integral boundary conditions are investigated. There is analyzed how spectrum of this problem depends on boundary condition parameters. Qualitative behaviour of all eigenvalues subject to nonlocal integral boundary condition parameters is described.

Key words: Sturm-Liouville problem, nonlocal integral condition

1. Introduction

Boundary problems with nonlocal conditions are a part of fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology and etc. Nonlocal conditions come up when value of the function on the boundary is connected to values inside the domain. Theoretical investigation of problems with various type of nonlocal boundary conditions is actual problem and recently it is paid much attention for them in the literature.

Originators of such problems were Samarskii and Bitsadze. They formulated and investigated nonlocal boundary problem for elliptic equation [1]. Canon was one of the pioneers who investigated parabolic problems with integral boundary conditions [3]. Also parabolic problems with nonlocal integral boundary conditions were analyzed in [2, 4, 5, 9, 12, 15, 16]. During the last decade the number of differential problems with nonlocal boundary conditions had increased.

Quite new area, related to problems of this type, deals with investigation of the spectrum of differential equations with nonlocal conditions. Eigenvalue problems with nonlocal conditions are closely linked to boundary problems for differential equations with nonlocal conditions [7, 8, 10]. In the papers [6, 11, 13, 14] the similar problems are investigated for the operators with nonlocal condition of Bitsadze-Samarskii or integral type.

The purpose of this paper is to analyze eigenvalue problem for stationary differential problem with two cases of nonlocal integral boundary conditions. We investigate how spectrum of this problem depends on some nonlocal boundary conditions parameters.

2. Sturm-Liouville Problem with Nonlocal Integral Boundary Condition

Let us consider the Sturm - Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad x \in (0,1), \tag{2.1}$$

$$u(0) = 0, \tag{2.2}$$

and another nonlocal integral boundary condition:

$$u(1) = \gamma \int_0^\xi u(x)\,\mathrm{d}x \quad (\text{ Case } 1), \tag{2.3}$$

or

$$u(1) = \gamma \int_\xi^1 u(x)\,\mathrm{d}x \quad (\text{ Case } 2), \tag{2.4}$$

with parameters $\gamma \in \bar{\mathbb{C}}$ and $\xi \in [0,1]$. In the general case eigenvalue $\lambda \in \mathbb{C}$ and eigenfunctions $u(x)$ are complex-valued functions. We investigate how spectrum depends on boundary condition parameters γ and ξ.

When $\gamma = 0$ or $\xi = 0$ problems (2.1)–(2.3) and (2.1)–(2.2), (2.4) are reduced to problems with classical boundary conditions. Their eigenvalues and eigenfunctions are well-known:

$$\lambda_k = (\pi k)^2, \quad u_k(x) = \sin(\pi k x), \quad k \in \mathbb{N}. \tag{2.5}$$

When $\lambda = 0$ then $u(x) = cx$. Substituting this solution into the second boundary condition we get

$$c = \gamma \int_0^\xi cx\,\mathrm{d}x = c\gamma\frac{\xi^2}{2} \quad (\text{ Case } \ 1), \quad c = \gamma \int_\xi^1 cx\,\mathrm{d}x = c\gamma\frac{1-\xi^2}{2} \quad (\text{ Case } \ 2).$$

Lemma 1. *The eigenvalue $\lambda = 0$ exists if and only if $\gamma = \dfrac{2}{\xi^2}$ in the Case 1 and $\gamma = \dfrac{2}{1-\xi^2}$ in the Case 2.*

In the general case, when $\lambda \neq 0$, eigenfunctions are $u = c\sin(qx)$ and eigenvalues $\lambda = q^2$, where $q \in \mathbb{C}_q \setminus \{0\}$, $\mathbb{C}_q := \{q \in \mathbb{C} | \operatorname{Re} q > 0$ or $\operatorname{Re} q = 0$, $\operatorname{Im} q > 0$ or $q = 0\}$. They satisfy equation (2.1), boundary condition (2.2) and nonlocal boundary condition (2.3) or nonlocal boundary condition (2.4).

When $\lambda \neq 0$ nonlocal boundary conditions are satisfied, if

$$c\,\sin(q) = c\,\gamma \int_0^\xi \sin(qx)\mathrm{d}x \quad (\text{ Case } \ 1), \tag{2.6}$$

$$c\,\sin(q) = c\,\gamma \int_\xi^1 \sin(qx)\mathrm{d}x \quad (\text{ Case } \ 2), \tag{2.7}$$

and nontrivial solution exists if q is a root of the equation

$$f_1(q) := 2\gamma \frac{\sin^2 \frac{\xi q}{2}}{q^2} - \frac{\sin q}{q} = 0 \quad (\text{ Case } 1), \tag{2.8}$$

$$f_2(q) := 2\gamma \frac{\sin(\frac{(1+\xi)q}{2})\sin(\frac{(1-\xi)q}{2})}{q^2} - \frac{\sin(q)}{q} = 0 \quad (\text{ Case } 2). \tag{2.9}$$

If $\sin q = 0$ and $\sin \frac{\xi q}{2} = 0$ in the first case, and $\sin(\frac{(1+\xi)q}{2}) = 0$ or $\sin(\frac{(1-\xi)q}{2}) = 0$ and $\sin(q) = 0$ in the second case, then equations (2.8) and (2.9) are valid for all $\gamma \in \mathbb{C}$. In this case we get *constant eigenvalues*, which don't depend on parameter γ. If parameter ξ is irrational number then such eigenvalues do not exist.

Let $\xi = r = \frac{m}{n} \in \mathbb{Q}$. For $\xi \in (0,1)$ we suppose that m and n $(n > m > 0)$ are positive coprime integer numbers. If $\xi = 0$ we suppose $m = 0$, $n = 1$ and if $\xi = 1$ we suppose $m = 1$, $n = 1$. Let denote subset $\mathbb{N}_m := \{n \in \mathbb{N} | n = km, k \in \mathbb{N}\}$ of integer positive numbers, $\mathbb{N}_e = \{k \in \mathbb{N}_2 | k \leq n\} \cup \{0\}$ – even numbers and $\mathbb{N}_o = \{k \in \mathbb{N} \setminus \mathbb{N}_2 | k \leq n\}$ – odd numbers.

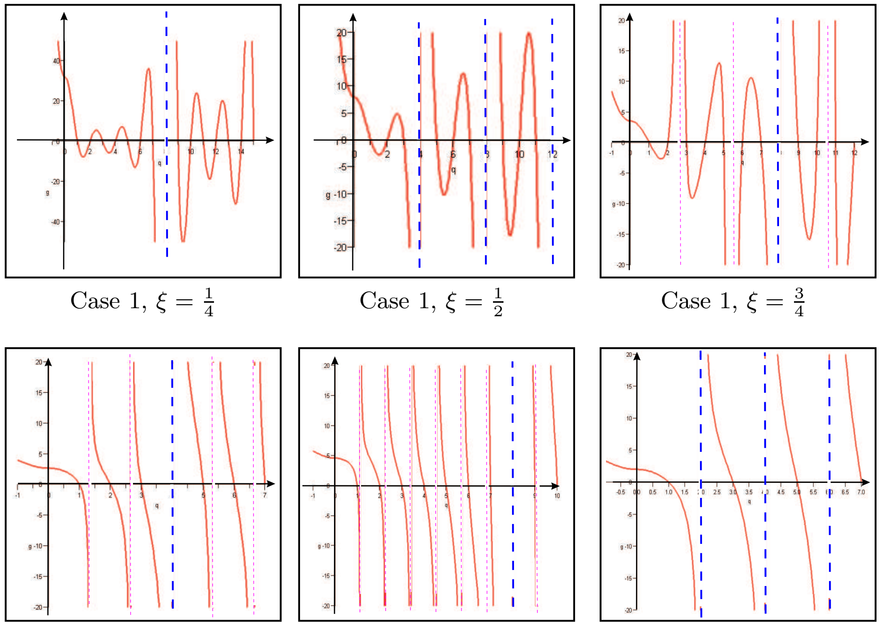
Lemma 2. *Constant eigenvalues exist only for rational* $\xi = \frac{m}{n} \in [0,1]$, *and those eigenvalues are equal:* $\lambda_k = (n\pi k)^2, k \in \mathbb{N}$, $m \in \mathbb{N}_e$ *and* $\lambda_k = (2n\pi k)^2, k \in \mathbb{N}$, $m \in \mathbb{N}_o$ *in Case 1;* $\lambda_k = (n\pi k)^2, k \in \mathbb{N}$, $n - m \in \mathbb{N}_e$ *and* $\lambda_k = (2n\pi k)^2, k \in \mathbb{N}$, $n - m \in \mathbb{N}_o$ *in Case 2.*

Lemma 3. *There is countable number of eigenvalues, which depend on parameter* γ. *They exist for every* $\gamma \in \overline{\mathbb{C}}$ *and every* $\xi \in (0,1]$ *in Case 1 and every* $\xi \in [0,1)$ *in Case 2. Point* $\lambda = \infty$ *is accumulation point of those eigenvalues.*

3. Real Eigenvalues Case

Now instead $q \in \mathbb{C}_q$ we take q only on rays $q = x \geq 0$ and $q = -ix$, $x \leq 0$. We have positive eigenvalues in the case $q = x > 0$ and negative eigenvalues in the case $q = -ix$, $x < 0$. Point $q = x = 0$ corresponds $\lambda = 0$.

For the real x we define functions:



Case 1, $\xi = \frac{1}{4}$ — Case 1, $\xi = \frac{1}{2}$ — Case 1, $\xi = \frac{3}{4}$

Case 2, $\xi = \frac{1}{2}$ — Case 2, $\xi = \frac{3}{4}$ — Case 1, $\xi = 1$; Case 2, $\xi = 0$

Figure 1. Functions $\gamma_1(x/\pi)$ and $\gamma_2(x/\pi)$.

$$\gamma_1(x) := \begin{cases} \gamma_{1-}(x) = \dfrac{x \sinh x}{2\sinh^2(\frac{\xi x}{2})}, & \text{for } x \le 0, \\ \gamma_{1+}(x) = \dfrac{x \sin x}{2\sin^2(\frac{\xi x}{2})}, & \text{for } x \ge 0; \end{cases} \tag{3.1}$$

$$\gamma_2(x) := \begin{cases} \gamma_{2-}(x) = \dfrac{x \sinh x}{2\sinh(\frac{(1+\xi)x}{2})\sinh(\frac{(1-\xi)x}{2})} & \text{for } x \le 0, \\ \gamma_{2+}(x) = \dfrac{x \sin x}{2\sin(\frac{(1+\xi)x}{2})\sin(\frac{(1-\xi)x}{2})} & \text{for } x \ge 0. \end{cases} \tag{3.2}$$

Function γ_{k+} corresponds to positive eigenvalues case, function γ_{k-} corresponds to negative eigenvalues case. Graphics of functions $\gamma_1(x)$ and $\gamma_2(x)$ for various ξ are shown in Fig. 1.

Few lemmas about real eigenvalues follow from the properties of those functions.

In the Sec. 1 we have shown that $\lambda = 0$ exists if and only if $\gamma = \gamma_0$ (see Lemma 1), and $\gamma_0 = \frac{2}{\xi^2}$ in Case 1, $\gamma_0 = \frac{2}{1-\xi^2}$ in Case 2.

Lemma 4. *For $\gamma > \gamma_0$ one negative eigenvalue exists, and for $\gamma \le \gamma_0$ there are no negative eigenvalues.*

We name points p_k in which functions γ_1 or γ_2 aren't defined, i.e. $\gamma_1(p_k) = \infty$ or $\gamma_2(p_k) = \infty$ as poles. We can enumerate all poles p_k, $k \in \mathbb{N}$ in the increasing order, i.e. $p_1 < p_2 < \cdots < p_k = p_{k+1} < \ldots$. There p_k is the first order pole witch consist with constant eigenvalue point. Formally we note $p_0 = 0$.

Lemma 5. *All eigenvalues of problem* (2.1)–(2.2), (2.4) *with real* γ *are real numbers. Each positive eigenvalue* $\lambda_k(\gamma) = x_k^2(\gamma)$, *where* $x_k \in (p_{k-1}, p_k)$, *if* $p_{k-1} < p_k$, *or* $x_k = p_k$, *if* $p_{k-1} = p_k$.

Corollary 1. For problem (2.1)–(2.2), (2.4) the following properties are valid:

$$\lim_{\gamma\to-\infty} x_k(\gamma) = p_k, \quad \lim_{\gamma\to+\infty} x_k(\gamma) = p_{k-1}, k \in \mathbb{N} \setminus \{1\}, \quad \lim_{\gamma\to+\infty} x_1(\gamma) = -\infty.$$

In Case 1 of the boundary conditions there isn't such simple spectrum. In this case for real γ multiple and complex eigenvalues can exist. In many cases it is necessary to know when all eigenvalues are positive and non multiple, i.e. the spectrum of problem with nonlocal conditions is similar to the spectrum of classical problems. When the qualitative root distribution depends on parameters γ and ξ, it is necessary to find such interval for γ where the spectrum of the problem satisfies this property.

Let us suppose that x_k, $k \in \mathbb{N}$ are positive roots of equation

$$\sin x + x\cos x = 0, \quad x_2 \approx 4.91318, \quad x_3 \approx 7.977.$$

Then we define

$$\xi_k := \frac{\pi}{x_k}, \quad \gamma_k := \frac{x_k \sin x_k}{2}, \quad k \in \mathbb{N},$$
$$\xi_2 \approx 0.639421, \quad \xi_3 \approx 0.393743, \quad \gamma_2 \approx -2.4072, \quad \gamma_3 \approx 3.95836.$$

Lemma 6. *If* $\gamma_2 \le \gamma \le \gamma_3$, *then all eigenvalues of the problem* (2.1)–(2.3) *are real for all* $\xi \in (0,1)$, *and limitary cases are realizable when* $\xi = \xi_2$ *and* $\xi = \xi_3$. *If* $\gamma_2 < \gamma \le 2$ *then all eigenvalues are positive and simple for all* $\xi \in (0,1)$.

4. Conclusions

1. Sturm-Liouville problems (2.1)–(2.3) (Case 1) and (2.1)–(2.2), (2.4) (Case 2) have similar spectrum properties in the complex plane. Both spectrums haven't constant eigenvalues for irrational ξ and countable number of non constant and constant eigenvalues for rational ξ. All constant eigenvalues are real positive numbers.
2. Both problems have only one negative eigenvalue for $\gamma > \gamma_0$.
3. Positive parts of spectrums are different for real γ case. Problem in Case 2 has only real eigenvalues. In Case 1 problem has all real eigenvalues only for $\gamma_m(\xi) \le \gamma \le \gamma_M(\xi)$, but exists interval $[\gamma_2, \gamma_3] \subset [\gamma_m, \gamma_M]$ the same for all ξ. So, in this case for some real γ multiple and complex eigenvalues exist.

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