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NUMERICAL METHODS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS WITH MIXED DERIVATIVES[1]

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Abstract. A class of singularly perturbed convection-diffusion problems is considered which contain a mixed derivative term. We consider the case when only regular layers appear in the solutions of problems from this transformed problem class. Under appropriate assumptions on the data of the problem we construct a decomposition of the solution into regular and layer components. We then introduce a numerical method on a piece-wise uniform fitted mesh that will generate approximations to solutions from the transformed problem class. We then show that in the perturbed case (i.e. when the perturbation parameter is small relative to the inverse of the number of mesh points) the approximations generated by the method converge uniformly with respect to the singular perturbation parameter. Finally numerical examples are presented that validate the theoretical result.

1. The Continuous Problem

Consider the following class of singularly perturbed elliptic convection-diffusion problems, posed on the unit square $\Omega = (0,1) \times (0,1)$:

$$L_\varepsilon u(x,y) \equiv (\varepsilon(au_{xx} + 2bu_{xy} + cu_{yy}) + \mathbf{a} \cdot \nabla u)(x,y) = f(x,y) \quad \text{in } \Omega, \quad (1.1a)$$

$$u(x,y) = 0 \quad \text{on } \partial\Omega, \quad (1.1b)$$

$$\mathbf{a}(x,y) = (a_1(x,y), a_2(x,y)) > (\alpha_1, \alpha_2) > (0,0), \qquad \forall (x,y) \in \overline{\Omega}, \qquad (1.1c)$$

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and the coefficients a, b and c satisfy the following ellipticity conditions: For all $(r,s) \in \mathbb{R}^2$

$$C_1(r^2+s^2) \le (ar^2+2brs+cs^2)(x,y) \le C_2(r^2+s^2), \quad (x,y) \in \Omega, \quad (1.1d)$$

where $C_1, C_2 > 0$ are positive constants.

The operator L_ε in (1.1a) satisfies the following minimum principle.

Lemma 1 [Minimum Principle]. *Let $v \in C^2(\overline{\Omega})$. If*

$$v(x,y) \ge 0, \quad \forall (x,y) \in \partial\Omega, \quad \textit{and} \quad L_\varepsilon v(x,y) \le 0, \quad \forall (x,y) \in \Omega,$$

then $v(x,y) \ge 0, \quad \forall (x,y) \in \overline{\Omega}$.

An immediate consequence of this is the following bound on the solution of problems from Problem Class 1.1.

$$\|u\| \le \frac{1}{\alpha}\, \|f\|, \qquad \text{where} \qquad \alpha = \max\{\alpha_1, \alpha_2\}$$

and $\|\cdot\|$ is the global maximum norm.

We now state some assumptions regarding the smoothness of the solutions of problems from Problem Class 1.1.

Assumption 1 *Assume that the functions a, b, c, a_1 and a_2 are smooth. Let $f \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. Assume that f satisfies the compatibility conditions*

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0. \quad (1.2)$$

Also, assume f is sufficiently regular and that the data of the problem satisfy additional compatibility conditions so that $u \in C^{3,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$.

Remark 1. With the previous assumption we are ruling out the existence of any corner singularities in the solutions of our problems. The local conditions (1.2) are sufficient in the case when $b \equiv 0$ (see [1].) Unfortunately for the Problem Class 1.1 it seems that such local conditions cannot be derived. This necessitates the introduction of this assumption as we require that $u \in C^{3,\nu}(\overline{\Omega})$ in our analysis.

We now give some classical bounds on the derivatives of problems from Problem Class 1.1.

Theorem 1. *Assume that a, b, c, a_1, a_2, $f \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. Let $u \in C^{3,\nu}(\overline{\Omega})$ be the solution of a problem from Problem Class 1.1. Then if $\|f\|_\nu \le C\varepsilon^{-1}$ we have*

$$|u|_k \le C\varepsilon^{-k}, \quad \textit{for} \quad k = 0,1,2,3.$$

2. Decomposition of Solution

The bounds on derivatives given in the previous section are not adequate for the analysis of our numerical method. In this section we establish sharper bounds on the derivatives by constructing a decomposition of the solution into regular and singular components. This is given by

$$u(x,y) = v(x,y) + w(x,y), \quad (x,y) \in \overline{\Omega}. \tag{2.1}$$

The regular component v is constructed via the consideration of a problem similar to that satisfied by u, but defined on an appropriate "extended" domain. Denote the solution of this problem by $v^\star$. Under suitable assumptions on the data of this problem we can show the following.

Theorem 2. *Let $\Omega^\star = (-d,1) \times (-d,1)$, where $d > 0$. Then $v^* \in C^{3,\nu}(\overline{\Omega}^*)$ and*

$$|v^*|_k \leq C(1 + \varepsilon^{2-k}) \quad \textit{for} \quad k = 0, \ldots, 3.$$

We then define v as $v = v^*|_{\overline{\Omega}}$, and the singular component thus satisfies the following homogeneous problem

$$L_\varepsilon w(x,y) = 0 \qquad \text{in} \ \ \Omega, \qquad w(x,y) = 0 \qquad \text{on} \ \ \partial\Omega_I, \tag{2.2a}$$

$$w(x,y) = -v(x,y) \quad \text{on} \ \ \partial\Omega_O, \tag{2.2b}$$

where

$$\Omega_I = \{(x,1) \mid 0 \leq x \leq 1\} \cup \{(1,y) \mid 0 \leq y \leq 1\}$$
$$\Omega_O = \{(x,0) \mid 0 \leq x \leq 1\} \cup \{(0,y) \mid 0 \leq y \leq 1\}.$$

We now give the required sharper bounds on the derivatives of w. Introduce the functions

$$A_1(x,y) = \frac{a_1(x,y)}{a(x,y)}, \qquad A_2(x,y) = \frac{a_2(x,y)}{c(x,y)}.$$

Theorem 3. *Let w be the solution of (2.2). Then w can be decomposed into the following sum*

$$w(x,y) = w_L(x,y) + w_B(x,y) + w_C(x,y), \quad (x,y) \in \overline{\Omega}, \tag{2.3}$$

where, for all $(x,y) \in \Omega$ we have the following bounds

$$|w_L(x,y)| \leq Ce^{-\gamma_1 x/2\varepsilon}, \ |w_B(x,y)| \leq Ce^{-\gamma_2 y/2\varepsilon}, \ |w_C(x,y)| \leq Ce^{-(\gamma_1 x+\gamma_2 y)/2\varepsilon},$$

and for all i,j, $\ 1 \leq i+j \leq 3$ we have

$$\left|\frac{\partial^{i+j} w_L}{\partial x^i \partial y^j}(x,y)\right| \leq C\varepsilon^{-i}(e^{-A_1(0,y)x/\varepsilon} + \varepsilon^{1-j}),$$

$$\left|\frac{\partial^{i+j} w_B}{\partial x^i \partial y^j}(x,y)\right| \leq C\varepsilon^{-j}(e^{-A_2(x,0)y/\varepsilon} + \varepsilon^{1-i}), \quad \left|\frac{\partial^{i+j} w_C}{\partial x^i \partial y^j}(x,y)\right| \leq C\varepsilon^{-(i+j)},$$

where

$$\gamma_1 = \min_{(x,y)\in\overline{\Omega}} \left\{\frac{a_1(x,y)}{a(x,y)}\right\}, \quad \gamma_2 = \min_{(x,y)\in\overline{\Omega}} \left\{\frac{a_2(x,y)}{c(x,y)}\right\}. \tag{2.4}$$

Remark 2. The key feature of the bounds on the derivatives of the layer components w_L and w_B, is that the magnitudes of the derivatives in the directions normal to the layers, have an extra positive power of ε.

3. Numerical Method

Our numerical method comprises an upwind finite difference operator on a fitted piecewise-uniform mesh. The difference operator L_ε^N, on a mesh Ω^N, is defined for any mesh function Z^N, as

$$L_\varepsilon^N Z_{i,j}^N \equiv \big(\varepsilon(a\delta_x^2+2(\tilde{b}^+\delta_{xy}^+ + \tilde{b}^-\delta_{xy}^-)+c\delta_y^2)+a_1D_x^+ + a_2D_y^+\big)Z_{i,j}^N, \quad \forall (x_i,y_j)\in\Omega^N,$$

where

$$\tilde{b}^\pm(x_i,y_j) = \begin{cases} 0.5(b(x_i,y_j)\pm|b(x_i,y_j)|), & i<N_x/2,\ j<N_y/2, \\ 0, & \text{otherwise}, \end{cases},$$

$$\delta_{xy}^+ Z_{i,j}^N = \frac{D_x^+D_y^+ + D_x^-D_y^-}{2}Z_{i,j}^N, \quad \delta_{xy}^- Z_{i,j}^N = \frac{D_x^+D_y^- + D_x^-D_y^+}{2}Z_{i,j}^N.$$

The operator L_ε^N is an inconsistent difference operator which only approximates the mixed derivative term in a subset of the mesh at which it is defined. We discretise the domain $\overline{\Omega}$ with the tensor product mesh $\overline{\Omega}_\sigma^N = \overline{\Omega}_{\sigma_1}^{N_x}\times\overline{\Omega}_{\sigma_2}^{N_y}$, where

$$\overline{\Omega}_{\sigma_1}^{N_x} = \{x_i \mid 0\le i\le N_x\}, \quad \text{and} \quad \overline{\Omega}_{\sigma_2}^{N_y} = \{y_j \mid 0\le j\le N_y\},$$

with

$$x_i = \begin{cases} 2i\sigma_1/N_x, & 0\le i\le N_x/2 \\ \sigma_1+2(i-N_x/2)(1-\sigma_1)/N_x, & N_x/2<i\le N_x \end{cases},$$

$$y_j = \begin{cases} 2j\sigma_2/N_y, & 0\le j\le N_y/2 \\ \sigma_2+2(j-N_y/2)(1-\sigma_2)/N_y, & N_y/2<j\le N_y \end{cases},$$

and

$$\sigma_1 = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\gamma_1}\ln(N_xN_y)\right\}, \quad \sigma_2 = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\gamma_2}\ln(N_xN_y)\right\}. \tag{3.1}$$

Setting $\partial\Omega_\sigma^N = \overline{\Omega}_\sigma^N\cap\partial\Omega$, the resulting fitted mesh finite difference method is

$$\begin{cases} L_\varepsilon^N U^N(x_i,y_j) = f(x_i,y_j) & \text{in } \Omega_\sigma^N, \\ U^N(x_i,y_j) = 0 & \text{on } \partial\Omega_\sigma^N. \end{cases} \tag{3.2}$$

The finite difference operator L_ε^N satisfies the following discrete minimum principle on $\overline{\Omega}_\sigma^N$.

Theorem 4 [Discrete Minimum Principle]. *Let Z^N be any mesh function defined on $\overline{\Omega}_\sigma^N$. If*

$$Z^N(x_i,y_j) \geq 0, \quad \forall (x_i,y_j) \in \partial\Omega_\sigma^N, \quad L_\varepsilon^N Z^N(x_i,y_j) \leq 0, \quad \forall (x_i,y_j) \in \Omega_\sigma^N,$$

and N_x, N_y satisfy the inequalities

$$\frac{|b_{i,j}|}{c_{i,j}} \leq \frac{\sigma_1 N_y}{\sigma_2 N_x} \leq \frac{a_{i,j}}{|b_{i,j}|}, \quad \forall i,j \text{ such that } \quad 0 < i < N_x/2, \quad 0 < j < N_y/2,$$

then $Z^N(x_i,y_j) \geq 0$, $\forall (x_i,y_j) \in \overline{\Omega}_\sigma^N$.

The following lemma gives a bound on the local truncation error of method (3.2).

Lemma 2 [Truncation Error]. *Let u be the solution of* (1.1) *and U^N be the solution of the discrete problem* (3.2) *defined on $\overline{\Omega}_\sigma^N$. Then the following gives a bound on the local truncation error in the corner mesh region*

$$\begin{aligned} \left|L_\varepsilon^N(U^N - u)(x_i,y_j)\right| \leq C \Bigg[\varepsilon \Bigg(h \left(\left\| \frac{\partial^3 u}{\partial x^3} \right\| + \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\| \right) \\ + \ k \left(\left\| \frac{\partial^3 u}{\partial y^3} \right\| + \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\| \right) \Bigg) + h \left\| \frac{\partial^2 u}{\partial x^2} \right\| + k \left\| \frac{\partial^2 u}{\partial y^2} \right\| \Bigg]. \end{aligned}$$

And the following expression gives a bound for the local truncation error in the remainder of the mesh

$$\begin{aligned} \left|L_\varepsilon^N(U^N - u)(x_i,y_j)\right| \leq C \Bigg[\varepsilon \Bigg((x_{i+1} - x_{i-1}) \left\| \frac{\partial^3 u}{\partial x^3} \right\| + (y_{j+1} - y_{j-1}) \left\| \frac{\partial^3 u}{\partial y^3} \right\| \\ + \|b\| \left\| \frac{\partial^2 u}{\partial x \partial y} \right\| \Bigg) + \ (x_{i+1} - x_i) \left\| \frac{\partial^2 u}{\partial x^2} \right\| + (y_{j+1} - y_j) \left\| \frac{\partial^2 u}{\partial y^2} \right\| \Bigg]. \end{aligned}$$

4. Decomposition of Numerical Solution and Error Estimates

In an analogous manner to the continuous case we decompose our numerical solution into a regular and singular component

$$U^N(x_i,y_j) = V^N(x_i,y_j) + W^N(x_i,y_j), \quad (x_i,y_j) \in \overline{\Omega}_\sigma^N,$$

where V^N is the solution of the inhomogeneous problem

$$\begin{cases} L_\varepsilon^N V^N(x_i,y_j) = f(x_i,y_j) & \text{in } \ \Omega_\sigma^N, \\ V^N(x_i,y_j) = v(x_i,y_j) & \text{on } \ \partial\Omega_\sigma^N, \end{cases} \tag{4.1}$$

and therefore W^N is the solution of the problem

$$\begin{cases} L_\varepsilon^N W^N(x_i,y_j) = 0 & \text{in } \Omega_\sigma^N, \\ W^N(x_i,y_j) = w(x_i,y_j) & \text{on } \partial\Omega_\sigma^N. \end{cases} \tag{4.2}$$

The error in our numerical solution can now also be decomposed

$$(U^N - u)(x_i,y_j) = ((V^N - v) + (W^N - w))(x_i,y_j), \quad (x_i,y_j) \in \overline{\Omega}_\sigma^N,$$

and the error in the regular component and the singular component can be estimated separately.

Assumption 2 *For convenience we set $N_x = N_y = N$ and assume that the hypotheses of Theorem 4 hold. We shall also assume that $\varepsilon \le N^{-1}$.*

As a consequence of this assumption we have the following

$$\sigma_1 = 2\frac{\varepsilon}{\gamma_1}\ln N, \quad \sigma_2 = 2\frac{\varepsilon}{\gamma_2}\ln N. \tag{4.3}$$

Theorem 5 [Error in the Regular Component]. *Under Assumption 2 the error in the smooth component satisfies the following ε-uniform error estimate*

$$\left|(V^N - v)(x_i,y_j)\right| \le CN^{-1}, \quad (x_i,y_j) \in \overline{\Omega}_\sigma^N.$$

Theorem 6 [Error in Singular Component]. *Under Assumption 2 the error in the singular component satisfies the following ε-uniform error estimate*

$$\left|(W^N - w)(x_i,y_j)\right| \le CN^{-1}(\ln N)^2, \quad (x_i,y_j) \in \overline{\Omega}_\sigma^N.$$

5. Numerical Results

To examine the performance of the method we let $N_x = N_y = N$ and tabulate the computed ε-uniform orders of convergence, p^N. These are calculated from the two-mesh differences defined as

$$D_\varepsilon^N = \max_{0\le i,j\le N} |U^N(x_i,y_j) - \overline{U}^{2N}(x_i,y_j)|, \quad D^N = \max_{\varepsilon=2^{-9},\dots,2^{-32}} D_\varepsilon^N,$$

where $\overline{U}^{2N}$ indicates the piecewise bilinear interpolant of the numerical solution U^N. The p^N are then defined as $p^N = \log_2 \dfrac{D^N}{D^{2N}}$.

Consider the following class of problems where m, α_1 and α_2 are constants and f is chosen so that conditions (1.2) are satisfied:

$$\begin{cases} (\varepsilon((1+m^2)u_{xx} + 2mu_{xy} + u_{yy}) + \alpha_1 u_x + \alpha_2 u_y)(x,y) = f(x,y) \text{ in } \Omega, \\ u(x,y) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We also define α_1 and α_2 so that the inequalities in Theorem 4 are satisfied.

For a range of values of m the ε-uniform orders of convergence are shown in Table 1 which indicates that our method performs uniformly well for all values of m considered.

	Number of Intervals N				
m	8	16	32	64	128
-3.00	0.86	0.80	0.76	0.83	0.84
-2.00	0.70	0.75	0.74	0.80	0.84
-1.00	0.53	0.63	0.67	0.73	0.79
0.00	0.53	0.64	0.68	0.74	0.80
1.00	0.70	0.75	0.74	0.80	0.84
2.00	0.83	0.76	0.76	0.82	0.84
3.00	0.87	0.81	0.77	0.83	0.83

Table 1. Values of p^N, for method (3.2) applied to Problem Class 5.1.

References

[1] H. Han and R.B. Kellogg. Differentiability properties of solutions of the equation $-\varepsilon^2 \Delta u + ru = f(x,y)$ in a square. *SIAM J. Math. Anal.*, **21**, 394–408, 1990.