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ON THE STABILITY OF WAKE FLOWS IN SHALLOW WATER[1]

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Abstract. Flows behind obstacles (such as islands) are shallow if the transverse scale of the flow is much larger than water depth. Experimental and theoretical analysis indicates that the development of shallow wakes is different from the wakes in deep water. Several authors have used the linear stability theory in order to understand when shallow flows become unstable. Two-dimensional depth-averaged Saint-Venant equations are usually used for the analysis. One of the main assumptions in shallow water theory is the independence of the velocity distribution on the vertical coordinate. In many cases, however, this assumption may not be valid. This paper presents an attempt to evaluate the influence of this assumption on the results of linear stability analysis of shallow wake flows with bottom friction. Momentum correction coefficients β_1 and β_2 in the x and y directions are used in order to take into account the non-uniformity of the velocity distribution in the vertical direction. It is shown that the stability boundary is quite sensitive to the variation of the parameters β_1 and β_2.

Key words: momentum correction coefficients, shallow wake flows

1. Introduction

Shallow wake flows are flows behind obstacles (such as islands) with the transverse scale of the flow being much larger than the vertical scale (water depth). Experiments show that development of wakes in shallow water significantly differs from the development of wakes in deep water. This is linked to the fact that limited water depth has a strong influence on the development of flow instabilities. Bottom friction acts as a suppression factor for the growth of

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transverse perturbations. Moreover, evolution of three-dimensional instabilities is prevented due to small vertical scale.

Vortex structures observed in shallow water in many cases may resemble flow patterns in deep water, but in shallow water case the corresponding flow patterns can be observed at much larger values of the Reynolds number. For example, photograph Nr. 173 by Van Dyke [2] shows formation of eddies organized into a vortex street behind an obstacle in shallow water although the Reynolds number for this case is 10^7 [2]. Note that vortex street pattern in unbounded flows is limited to significantly smaller Reynolds numbers.

Several authors analyzed the stability of shallow flows both experimentally and theoretically [1, 3, 4, 8]. One of the main assumptions which is usually made in shallow water theory in order to facilitate the analysis is the independence of the flow characteristics on the vertical coordinate since shallow water equations are depth-averaged equations. There are many cases, however, where this assumption may not be valid. Changes in flow geometry, flow regimes or roughness of the bottom boundary can lead to large deviations from the above-mentioned assumption [9, 10]. Momentum correction coefficients are applied by several authors [9, 10] in order to take into account the non-uniformity of the velocity distribution. In particular, momentum correction coefficients are used in [6] for linear stability analysis of shallow mixing layers.

The present paper makes an attempt to evaluate the importance of the non-uniformity of the velocity distribution on the stability analysis of shallow wake flows. Momentum correction coefficients are used in this paper to calculate the stability boundary of the flow for the following wake profile

$$U(y) = 1 + \frac{2R}{1-R}\frac{1}{\cosh^2(\alpha y)}.$$

2. Problem Formulation

The governing equations for shallow flow under the rigid-lid assumption are [10]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + (2\beta_1 - 1)u\frac{\partial u}{\partial x} + (\beta_2 - 1)u\frac{\partial v}{\partial y} + \beta_2 v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \frac{c_f}{2h}u\sqrt{u^2+v^2}, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + (\beta_2 - 1)v\frac{\partial u}{\partial x} + \beta_2 u\frac{\partial v}{\partial x} + (2\beta_3 - 1)v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - \frac{c_f}{2h}v\sqrt{u^2+v^2}, \tag{2.3}$$

where x and y are the spatial coordinates, t is the time, u and v are the depth-averaged velocity components in the x and y directions respectively, h

is water depth, c_f is the friction coefficient defined by the equation [5]:

$$\frac{1}{\sqrt{c_f}} = -4\log(\frac{1.25}{4Re\sqrt{c_f}}),$$

where Re is the Reynolds number.

Shear stress at the boundary is modeled by the Chezy formula

$$\tau_{wx} = \frac{1}{2}c_f \rho u\sqrt{u^2+v^2}, \quad \tau_{wy} = \frac{1}{2}c_f \rho v\sqrt{u^2+v^2},$$

where ρ is density, τ_{wx} and τ_{wy} are wall shear stresses along the x and y directions respectively.

The coefficients β_1, β_2, and β_3 in equations (2.1) – (2.3) are the momentum correction coefficients which are introduced in order to take into account non-uniformity of velocity distribution in the vertical direction. The momentum correction coefficients are defined as follows:

$$\beta_1 = \frac{1}{hu^2}\int_{z_1}^{z_2} \tilde{u}^2\,dz, \quad \beta_2 = \frac{1}{huv}\int_{z_1}^{z_2} \tilde{u}\tilde{v}\,dz, \quad \beta_3 = \frac{1}{hv^2}\int_{z_1}^{z_2} \tilde{v}^2\,dz,$$

where $\tilde{u}$ and $\tilde{v}$ are the velocity components in the x and y directions respectively. It is assumed that the coefficients β_1, β_2 and β_3 are independent on the spatial coordinates x and y.

Introducing the stream function $\psi(x, y, t)$ defined by the relations

$$u = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$$

and eliminating the pressure p we rewrite the equations (2.1)-(2.3) in the following form:

$$\begin{aligned}(\Delta\psi)_t &+ (2\beta_1-\beta_2)(\psi_y\psi_{xy})_y - \beta_2(\psi_x\psi_{yy})_y + (\beta_2-1)(\psi_x\psi_{xy})_x \\ &+ \beta_2(\psi_{xx}\psi_y)_x - (2\beta_3-1)(\psi_x\psi_{xx})_y + \frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} \\ &+ \frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) = 0, \qquad (2.4)\end{aligned}$$

where Δ is the Laplacian in two dimensions and the subscripts indicate the derivatives with respect to the variables x and y.

Suppose that the base flow

$$\boldsymbol{U} = (U(y), 0) \qquad (2.5)$$

is perturbed and the perturbed solution to the equation (2.4) is assumed to be of the form

$$\psi = \psi_0 + \epsilon\psi_1 + ... \qquad (2.6)$$

where ϵ is a small parameter and $\psi_{0y} = U$. Substituting (2.5) and (2.6) into (2.4) and linearizing the resulting equation in the neighborhood of the base flow (2.5) we obtain

$$\psi_{1xxt} + \psi_{1yyt} + (2\beta_1 - \beta_2)(U_y\psi_{1xy} + U\psi_{1xyy}) - \beta_2(U_y\psi_{1xy} + U_{yy}\psi_{1x}) + \beta_2 U\psi_{1xxx} + \frac{c_f}{2h}(U\psi_{1xx} + 2U_y\psi_{1y} + 2U\psi_{1yy}) = 0. \quad (2.7)$$

According to the method of normal modes we seek the perturbed component ψ_1 of the stream function in the form

$$\psi_1(x, y, t) = \phi_1(y)e^{ik(x-ct)} + c.c. \quad (2.8)$$

where k is a wavenumber and $c = c_r + ic_i$ is a complex eigenvalue, "c.c." means "complex conjugate". Substituting (2.8) into (2.7) we obtain the linearized stability equation (the modified Rayleigh equation) in the form:

$$\phi_1''\left[(2\beta_1 - \beta_2)U - c + \frac{c_f}{ikh}U\right] + U_y(2\beta_1 - 2\beta_2 + \frac{c_f}{ikh})\phi_1' + (k^2c - \beta_2 U_{yy} - k^2\beta_2 U - \frac{c_f}{2ih}kU)\phi_1 = 0 \quad (2.9)$$

with the boundary conditions

$$\phi_1(\pm\infty) = 0. \quad (2.10)$$

3. Solution Method

It is known that for unbounded flows the spectrum consists of both a discrete an a continuous part [7]. As discussed in [7], for practical and computational purposes it is often possible to use simpler formulation where a discretized approximation of the continuous spectrum is used. Therefore only a discrete spectrum is analyzed in the present paper.

Using the substitution

$$x = \frac{2}{\pi}\arctan(y), \quad y \in (-\infty; +\infty), \ x \in [-1; 1],$$

we seek the solution $\phi(x)$ of the modified Rayleigh equation in the form:

$$\phi_1(x) = \sum_{k=0}^{N-1} a_k(1 - x^2)T_k(x), \quad (3.1)$$

where a_k are unknown constants, and $T_k(x)$ is an n-order Chebyshev polynomial that has the form $T_k(x) = \cos(k * \arccos(x))$. The multiplier $(1 - x^2)$ is used to satisfy the boundary conditions (2.10) at $x = \pm 1$. Using the collocation method and choosing the points $x_j = cos(\pi j/(N + 1))$ as the collocation points we obtain the generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0, \tag{3.2}$$

where A and B are two complex-valued matrices and $a = (a_0 a_1 ... a_{N-1})^T$.

Solving the generalized eigenvalue problem (3.2), for given c_f and k we obtain a set of eigenvalues c_m. The imaginary parts c_{im} of eigenvalues $c_m = c_{rm} + ic_{im}$ determine linear stability of the base flow. The flow is said to be linearly stable if the imaginary parts of all c_m are negative. If the imaginary part of the eigenvalue c_m of at least one mode is positive then a perturbation grows exponentially with time and the flow is said to be linearly unstable.

Calculations show that for sufficiently large values of the friction coefficient c_f all eigenvalues have negative imaginary parts ($c_{im} < 0$), so the flow is stable. By decreasing c_f for given k it is possible to reach the point where at least one c_{im} becomes positive and the flow loses stability. The bisection method enables us to find the value of friction coefficient c_f for which at least one c_{im} is close to zero, while all other c_{im} are negative. This point lies on the "border" between the stability and the instability region of the flow. By repeating the process for different values of the wavenumber k we are able to build a neutral stability curve that is defined as a set of points in the (k, c_f)-plane for which one c_m has the imaginary part equal to zero, while imaginary parts of all other c_m are negative. The neutral stability curve represents the boundary separating the stability domain (above the curve) from the instability domain (below the curve). The critical value, $c_f^{(c)}$ of the parameter c_f is defined as the coordinate of the highest point of the curve, or $c_f^{(c)} = \max_k(c_f(k))$.

4. Results and Discussion

This paper presents an attempt to evaluate the influence of momentum correction coefficients on the value of the $c_f^{(c)}$ parameter. The influence is evaluated by solving problems (2.9) – (2.10) for different values of momentum correction coefficients β_1 and β_2, and comparing the critical values, $c_f^{(c)}$, of the parameter c_f. The linear stability results are presented for the following wake profile (see [1] for the definition of R and α):

$$U(y) = 1 + \frac{2R}{1-R}\frac{1}{\cosh^2(\alpha y)}.$$

The values of $c_f^{(c)}$ have been calculated for the following values of the parameters

$$\beta_1 = 1.00,\ 1.05,\ 1.10, \quad \beta_2 = 1.00,\ 1.05,\ 1.10.$$

The value of R is fixed at $R = -0.5$.

Figure 1 presents results of the comparison of the $c_f^{(c)}$ parameter calculated for different values of momentum correction coefficients β_1 and β_2. The results are compared to the values of $c_f^{(c)}$ calculated for β_1=1.00 and β_2=1.00 that corresponds to the case when the velocity non-uniformity across the vertical coordinate is not taken into account. As it can be seen, for some combinations

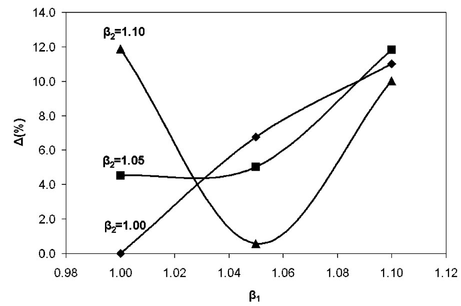

Figure 1. The percentage difference Δ between the values of the $c_f^{(c)}$ for depth-averaged equations ($\beta_1 = 1$, $\beta_2 = 1$) and equations with correction factors ($\beta_1 > 1$, $\beta_2 > 1$).

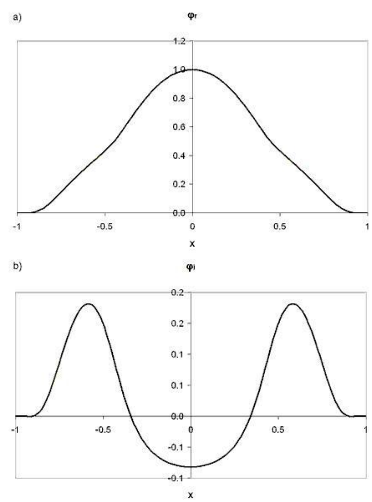

Figure 2. The real (a) and imaginary (b) parts of an eigenfunction obtained at $\beta_1 = 1$, $\beta_2 = 1$ and $R = -0.9$.

of the values of β_1 and β_2 the relative error can reach 10 %. The increase of β_1 leads to growth of $c_f^{(c)}$, so the flow becomes more unstable. The β_2 coefficient has, in turn, stabilizing effect on the flow, but its influence diminishes with the growth of β_1. The real and imaginary parts of the eigenfunction, $\phi(x) = \phi_r(x) + i\phi_i(x)$, are shown in Fig. 2 for $R = -0.9$ and $\beta_1 = \beta_2 = 1.00$.

Unfortunately, the values of coefficients β_1 and β_2 for real island wakes are not known. However as the error in determining the $c_f^{(c)}$ parameter may grow with increased values of β_1 (the stability boundary can be underestimated with increase of β_1) it might be important to know the values of β_1 and β_2 for the analyzed shallow water flows.

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