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NONLINEAR DYNAMICS OF BUBBLE INTERFACE IN VERTICAL HELE-SHAW CELL WITH MAGNETIC LIQUID UNDER THE ACTION OF NORMAL MAGNETIC FIELD

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Abstract. Dynamics of the interface of the bubble rising in the vertical Hele-Shaw cell with magnetic liquid under the action of normal field is studied. The nonlinear system of ODEs describing the interface dynamics of bubble is derived by the method of conformal mapping. Some useful for numerical calculations properties of the system of ODEs are described.

Key words: the Hele-Shaw cell, free interface, magnetic liquid,conformal maps

1. Introduction

The free surface flows in Hele-Shaw cells have obtained a great interest recently [2]. This is caused by extremely rich pattern formation phenomena in these systems. These phenomena obtains new features if one of the liquids is magnetic, when due to self-magnetic field forces intricate labyrinthine patterns are formed [5, 6, 9]. The problem of the free interface of bubble moving in the Hele-Show cell was considered in pioneering work [12] by Soffman and Taylor. Rather efficient approach to the investigation of free interface dynamics in Hele-Shaw cells is based on method of conformal mapping. By conformal mapping method the time evolution of the free interface leading to the formation of the Saffman-Taylor finger [2], finite time interface singularity formation due to the dynamics of the poles of mapping function [11] and nonlinear noise induced instability of viscous fingers at large capillary numbers

[1] are studied. Among later developments of the conformal mapping method we can mention the establishment of the 2D Toda hierarchy describing the interface evolution in the case of circular Saffman-Taylor instability when the surface tension is absent [8]. Finger shape selection problem based on exact time dependent solutions for conformal maps is considered in [7] (see also the discussion given in [10]).

In this paper the first time the conformal mapping method is developed for the dynamics of bubbles rising in the vertical Hele-Shaw cell with magnetic liquid. It allows one to follow in natural way the complex dynamics of the bubble interface. The interface evolution equation in terms of the conformal mapping function is obtained and it is shown that the interface dynamics can be formulated in terms of nonlinear ODEs for the coefficients of the Laurent series. An efficient numerical algorithm for numerical solution of the system of ODEs is developed.

2. Model and Mathematical Formulation of the Problem

Magnetic liquid motion surrounding the bubble in the Hele-Shaw cell is described by Darcy equation taking into account the gravitational and magnetic forces [3, 4]:

$$-\nabla p - \alpha\vec{v} + \frac{2M}{h_0}\nabla\varphi_0 + \rho\vec{g} = 0; \quad \mathbf{div}\,\vec{v} = 0. \tag{2.1}$$

Here $\alpha = 12\eta/h_0^2$ is the friction coefficient of the liquid in the Hele-Shaw cell with thickness h_0, φ_0 is the value of the magnetostatic potential of the liquid on the boundary of the Hele-Shaw cell:

$$\varphi_0 = -M\iint_D \Big[\frac{1}{|\vec{\rho}-\vec{\rho}'|} - \frac{1}{\sqrt{(\vec{\rho}-\vec{\rho}')^2 + h_0^2}}\Big]dS', \tag{2.2}$$

where D is a domain outside of the bubble. The boundary condition for the problem (2.1) – (2.2) on a bubble interface Σ is given by the Laplace law

$$p|_{\Sigma} = p_0 - \sigma k, \quad p_0 = const.$$

Here k is the curvature of the bubble interface, σ is coefficient of the surface tension. Evolution of interface in time is given by kinematic boundary condition for velocity $\vec{v}$ on Σ:

$$\alpha\vec{n}\cdot\vec{v} = \vec{n}\cdot(-\nabla p + \frac{2M}{h_0}\nabla\varphi_0 + \rho\vec{g}). \tag{2.3}$$

Here $\vec{n}$ is the normal to the interface. The condition of the incompressibility of the bubble is given by $S_b = \pi R^2, \;\; R = const$. The problem is completely formulated when the initial configuration of the bubble interface Σ is specified at the moment $t = 0$.

The solution of the problem (2.1) – (2.3) is determined by 3 dimensionless parameters: the dimensionless thickness of Hele-Shaw cell h, the gravitational

Bond number Bg and the magnetic Bond number Bm which are defined as follows

$$h = \frac{h_0}{R},\ Bg = \frac{\rho g R^2}{\sigma},\ Bm = \frac{2M^2 h_0}{\sigma}. \tag{2.4}$$

To put equations in dimensionless form the following scales are introduced: time t is scaled by $12\eta R^3/(\sigma h_0^2)$ – the characteristic capillary relaxation time of the bubble in the Hele-Shaw cell, the distances by the radius of bubble R defined by relation (2.4), and the pressure by a characteristic capillary pressure σ/R. As a result the problem in dimensionless form reads

$$-\nabla p - \vec{v} + Bm\nabla\varphi_0/h^2 - Bg\,\vec{e}_x = 0;\ \mathbf{div}\,\vec{v} = 0, \tag{2.5}$$

$$\varphi_0 = -\iint_D \Big[\frac{1}{|\vec{\rho} - \vec{\rho}'|} - \frac{1}{\sqrt{(\vec{\rho} - \vec{\rho}')^2 + h^2}}\Big] dS', \tag{2.6}$$

$$p|_\Sigma = p_0 - k, \quad S_b = \pi, \tag{2.7}$$

$$\vec{n}\cdot\vec{v}|_\Sigma = \vec{n}\cdot(-\nabla p + Bm\nabla\varphi_0/h^2 - Bg\vec{e}_x)|_\Sigma. \tag{2.8}$$

3. Conformal Mapping Method

The conformal mapping method for the bubble interface dynamics is introduced as follows. Let us denote the physical plane of bubble motion as plane of complex variable z. Let the function $z = f(w,t)$ be the conformal map of the domain outside the unit circle $|w| > 1$ to the domain outside the bubble. For uniqueness of the map we require $f(\infty,t) = \infty,\ f'_w(\infty,t) = c_{-1}(t) > 0$, where $c_{-1}(t)$ is given function. Representing $f(w,t)$ by the Laurent series in the domain $|w| > 1$ we have

$$z = f(w,t) = c_{-1}(t)w + \sum_{k=0}^{\infty} c_k(t) w^{-k}. \tag{3.1}$$

The parametric equation of the bubble interface $z = f(e^{i\theta},t)$ is given as

$$z = c_{-1}(t)e^{i\theta} + \sum_{k=0}^{\infty} c_k(t)\exp(-ik\theta),\ \ -\pi \le \theta \le \pi. \tag{3.2}$$

Calculating the area of the bubble according to

$$S_b = \frac{1}{2}\,\mathrm{Im}\int_{-\pi}^{\pi} \overline{f(e^{i\theta},t)}\frac{d}{d\theta} f(e^{i\theta},t)d\theta = \pi, \tag{3.3}$$

we obtain

$$c_{-1}^2(t) = 1 + \sum_{k=1}^{\infty} k|c_k(t)|^2. \tag{3.4}$$

Thus a positive function $c_{-1}(t)$ is determined by volume conservation condition (2.7). Relations (3.2) and (3.4) show that interface dynamics is determined by a set of time dependent functions $c_n(t)$. The set of ODE for these functions is derived as follows. From Darcy equation (2.5), when effective pressure $\tilde{p} = p - Bm/(h^2)\varphi_0$ and complex potentials for the vectorial fields in their complex representation are introduced: $\vec{v} \sim \overline{F'(z)}, \nabla\tilde{p} \sim \overline{\Phi'(z)}$ and $\vec{e}_x \sim 1$, we obtain

$$F'(z) + \Phi'(z) + Bg = 0, \quad z \in D. \tag{3.5}$$

Here F and Φ are analytical functions in the domain D. Using

$$z = f(w,t);\ F_1(w,t) := F(f(w,t));\ \Phi_1(w,t) := \Phi(f(w,t));\ \frac{dz}{dw} = f'_w(w,t)$$

the following equation is obtained

$$F_1(w,t) + \Phi_1(w,t) + Bg\, f(w,t) = C(t). \tag{3.6}$$

Now we exclude F_1 and Φ_1 from (3.6) at $w = e^{i\theta}$ using (2.5) – (2.8), and several conversions with complex functions. Then we obtain the nonlinear equation for the function $g(e^{i\theta},t) := Bg\,t + f(e^{i\theta},t)$:

$$\frac{\partial}{\partial t}\, g(e^{i\theta},t) = \frac{\partial}{\partial\theta}\, g(e^{i\theta},t)\big[\mathbf{H}[L(\theta,t)] - iL(\theta,t)\big], \tag{3.7}$$

where $\mathbf{H}$ is Hilbert transformation

$$\mathbf{H}[f(\theta)] = v.p.\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\cot\frac{\tau-\theta}{2}d\tau, \tag{3.8}$$

$$L(\theta,t) = \frac{2Bg\,c_{-1}(t)\cos\theta + \mathbf{H}[\mu'(\theta)]}{|c_{-1}(t) - \sum\limits_{k=1}^{\infty} kc_k(t)\exp[-i(k+1)\theta]|^2}, \tag{3.9}$$

$$\mu(\theta,t) = k(\theta,t) + Bg\varphi(\theta,t)/h^2;\ \varphi - \varphi_0 = const. \tag{3.10}$$

4. Reduction to the Cauchy Problem for ODE

To simplify notations we introduce $c_k(t) = c_k;\ c'_k(t) = \dot{c}_k;\ g(e^{i\theta},t) = g(e^{i\theta})$, $\mu(\theta,t) = \mu(\theta)$ and

$$Q(\theta) := \sum_{k=1}^{\infty} kc_k \exp\left[-i(k+1)\theta\right]; \tag{4.1}$$

$$R(\theta,\tau) := \frac{g(e^{i\theta}) - g(e^{i\tau})}{e^{i\theta} - e^{i\tau}};\ R(\tau,\tau) = c_{-1} - Q(\tau); \tag{4.2}$$

$$T(\theta,\tau) := \frac{[c_{-1} - Q(\tau)]\exp\left(i(\tau-\theta)/2\right)}{R(\theta,\tau)}. \tag{4.3}$$

From equalities $|g(e^{i\theta}) - g(e^{i\tau})| = 2|R(\theta,\tau)||\sin\left((\theta-\tau)/2\right)|$; $e^{i\tau}|e^{i\theta} - e^{i\tau}|/(e^{i\theta} - e^{i\tau}) = i\exp\left[i(\tau-\theta)/2\right]\mathrm{sgn}(\sin\left((\tau-\theta)/2\right))$ after some transformations we obtain

$$\varphi(\theta) = 2h \int_{-\pi}^{\pi} P(\theta,\tau)\mathrm{Im}[\mathrm{T}(\theta,\tau)]\mathrm{d}\tau, \tag{4.4}$$

where the real function P is given as follows

$$P(\theta,\tau) = \frac{|R(\theta,\tau)|\mathrm{sgn}(\sin\frac{\tau-\theta}{2})}{2|\sin\frac{\tau-\theta}{2}||R(\theta,\tau)| + h + \sqrt{4\sin^2\frac{\tau-\theta}{2}|R(\theta,\tau)|^2 + h^2}}. \tag{4.5}$$

As a result for effective capillary pressure $\mu(\theta) = k(\theta) + Bm/(h^2)\,\varphi(\theta)$ we obtain

$$\mu(\theta) = \frac{1 - \mathrm{Im}[\mathrm{Q}'(\theta)/(\mathrm{c}_{-1} - \mathrm{Q}(\theta))]}{|c_{-1} - Q(\theta)|} + \frac{2Bm}{h}\int_{-\pi}^{\pi} P(\theta,\tau)\mathrm{Im}[\mathrm{T}(\theta,\tau)]\mathrm{d}\tau. \tag{4.6}$$

Let us introduce real coefficients

$$\hat{a}_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\mu(\theta)\cos k\theta d\theta,\ \hat{b}_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin k\theta\mu(\theta)d\theta,\ k = 0,1,\ldots. \tag{4.7}$$

Then the real function

$$L(\theta) = \frac{2Bg\,c_{-1}\cos\theta - \sum_{k=1}^{\infty} k(\hat{a}_k\cos k\theta + \hat{b}_k\sin k\theta)}{|c_{-1} - Q(\theta)|^2}, \tag{4.8}$$

defined by (3.9) and (4.8) has complex Fourier coefficients

$$A_j = \frac{1}{\pi}\int_{-\pi}^{\pi} L(\theta)\exp(ij\theta)d\theta,\ j = 0,1,2,\ldots \tag{4.9}$$

The equation (3.7) reads

$$\dot{c}_{-1} + \sum_{k=0}^{\infty}\dot{c}_k e^{-i(k+1)\theta} = \big(A_0/2 + \sum_{j=1}^{\infty} A_j e^{-ij\theta}\big)\big(c_{-1} - \sum_{k=1}^{\infty} kc_k e^{-i(k+1)\theta}\big). \tag{4.10}$$

Comparing the coefficients at $\exp(-im\theta)$ for different m gives the system of ODEs:

$$\dot{c}_0 = c_{-1}A_1;\ \ \dot{c}_k = c_{-1}A_{k+1} - A_0 kc_k/2 - \sum_{j=1}^{k-1} A_j(k-j)c_{k-j}, \quad k \geq 1, \tag{4.11}$$

where $c_{-1}^2 = 1 + \sum_{k=1}^{\infty} k|c_k|^2$. A_j due to (4.1)–(4.9) depend on c_k only. Initial values $c_k(0)$ are determined by shape of a bubble at $t = 0$:

$$z = [c_{-1}(0) + \sum_{k=0}^{\infty} c_k(0)\exp(-i(k+1)\theta)]\exp(i\theta)$$

and must be to added to (4.11) for uniqueness of the solution.

For functions $A_j,\ j = 0,1,\ldots$, which depend on $c_k,\ k \geq 0$, and make up the ODEs (4.11) another useful representation may be obtained.

Proposition 1. *Let us define the Fourier coefficients*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos k\tau d\tau}{|c_{-1} - Q(\tau)|^2};\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin k\tau d\tau}{|c_{-1} - Q(\tau)|^2},\ k = 0, 1, \dots, \quad (4.12)$$

which depend on c_j*. Then functions* $B_j(s) := \dfrac{d}{ds}\mathbf{H}\Big[\dfrac{e^{ijs}}{|c_{-1} - Q(s)|^2}\Big]$ *have the following representation*

$$B_j(s) = -\frac{\exp(ijs)}{2}\Big\{a_0 j + \sum_{k=1}^{\infty}[(a_k - ib_k)(k+j)e^{iks} + (a_k + ib_k)|j-k|e^{-iks}]\Big\}, \quad (4.13)$$

and $A_j = Bg\, c_{-1} I_j + \frac{1}{\pi}\int_{-\pi}^{\pi} \mu(s)B_j(s)\, ds$*, where* $I_0 = 2a_1$*;* $I_j = a_{j-1} + ib_{j-1} + a_{j+1} + ib_{j+1}$*, if* $j = 1, 2, \dots$*, and* $\mu(s)$ *is defined by (4.6).*

Proposition 2. *Let be* $D_j(s) = \mathbf{H}\Big[\dfrac{e^{ijs}}{|c_{-1} - Q(s)|^2}\Big] = \int B_j(s)\, ds$*. Then*

$$A_j = Bgc_{-1}I_j - \frac{1}{\pi}\int_{-\pi}^{\pi} D_j(s)\mu'(s)ds. \quad (4.14)$$

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