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ON INTEGRAL EQUATIONS DRIVEN BY P-VARIATION FUNCTIONS AND PROCESSES

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Abstract. A brief introduction to the problems of integration is given under processes with the bounded p-variation. The existence and asymptotic behavior of the approximation of the unique strong solution of SIEs driven by a special continuous p-semimartingale are discussed.

Key words: p-variation, p-semimartingales, stochastic integrals

In the last few years, a fractional Brownian motion (fBm) has been the subject of numerous investigations. The fBm with the Hurst index $0 < H < 1$ is a centered Gaussian process $X = \{X_t, t \geqslant 0\}$ with $X_0 = 0$ and covariance

$$\mathrm{Cov}\big(X_t, X_s\big) = \frac{1}{2}\,\mathrm{Var}(X_1)\big(t^{2H} + s^{2H} - |t-s|^{2H}\big)$$

for all $t, s \geqslant 0$. If $\mathrm{Var}(X_1) = 1$, we write $X = B^H$. The case $H = 1/2$ corresponds to the standard Brownian motion.

This process has stationary increments, self-similarity and long-range dependence properties. These properties make the fBm a suitable driving noise in different applications such as mathematical finance and network traffic analysis. From the theoretical point of view, it is an interesting process because it is neither a Markov process nor a semimartingale (if $H \neq 1/2$). So we cannot use well-developed theories. In oder to develop these applications, one needs to construct a stochastic calculus with respect to the fBm. The different definitions of stochastic integrals with respect to B^H can be sorted into two main groups:

- those which rely on the sample-path properties of B^H,
- those based on its Gaussianity.

We do not focus our attention on the Gaussianity of B^H.

The p-variation, $0 < p < \infty$, of a real-valued function f on $[a,b]$ is defined as

$$v_p(f;[a,b]) = \sup_{\varkappa} \sum_{k=1}^{n} \bigl|f(x_k) - f(x_{k-1})\bigr|^p,$$

where the supremum is taken over all subdivisions $\varkappa = \{x_i : i = 0,\dots,n\}$ of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. If $v_p(f;[a,b]) < \infty$, f is said to have a bounded p-variation on $[a,b]$. Let

$$\mathcal{W}_p([a,b]) := \bigl\{f\colon\ [a,b] \to \mathbb{R} : v_p(f;[a,b]) < \infty\bigr\}.$$

We recall that the fBm is locally Hölder continuous of order α for every $\alpha < H$. Thus the fBm has a locally bounded p-variation for $p > 1/H$, i.e. for almost all ω, $v_p(B^H(\cdot,\omega);[0,T]) < \infty$ for all $T > 0$.

Since Young's paper [17] on the Stieltjes integration, it has been known that the Riemann-Stieltjes integral may exist even if both integrand and integrator have unbounded variation.

Theorem 1. (see [2, 17].) *Assume $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ for some p, $q > 0$ with $1/p + 1/q > 1$. Then the integral $\int_a^b f\,\mathrm{d}h$ exists*

i) as the Riemann-Stieltjes (RS) integral if f and h have no common discontinuities;

ii) as the refinement Riemann-Stieltjes (RRS) integral if f and h have no common discontinuities on the same side at the same point. In particular, this necessary condition is satisfied if f is left-continuous and h is right-continuous or vice versa.

In whichever of the two senses the integral exists, for any $y \in [a,b]$, the Love–Young inequality

$$\left| \int_a^b f\,dh - f(y)\bigl[h(b) - h(a)\bigr] \right| \le C_{p,q} V_q(f;[a,b]) V_p(h;[a,b])$$

holds, where

$$V_q(\,\cdot\,;[a,b]) = v_q^{1/q}(\,\cdot\,;[a,b]), \quad C_{p,q} = \zeta(p^{-1} + q^{-1}), \quad \zeta(s) = \sum_{n\ge1} n^{-s}.$$

Remark 1. The integral $\int_a^b f dh$ exists as the RS integral if and only if it exists as the RRS integral and the two functions f, h have no common discontinuities on $[a,b]$.

Freedman [4] was the first to concern with the RS type linear integral equations driven by a continuous function of bounded p-variation with $1 \leqslant p < 2$. T. Lyons [13] extended [4] result for nonlinear integral equations. He considered the integral equation

$$y_t = a + (RS)\int_0^t f(y_s)\,dh_s, \qquad 0 \leqslant t \leqslant T, \tag{0.1}$$

where $h \in C\mathcal{W}_p([0,T])$, i.e. h is continuous and has a bounded p-variation on $[0,T]$ for some p, $1 \leqslant p < 2$. Lyons has proved that this equation has a unique solution in the space $C\mathcal{W}_p([0,T])$ if $f \in C^{1+\alpha}(\mathbb{R})$ for some $\alpha > p-1$.

For $0 < \alpha \leqslant 1$, $C^{1+\alpha}(\mathbb{R})$ denote the set of all C^1-functions $g\colon\ \mathbb{R} \to \mathbb{R}$ such that

$$\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x-y|^\alpha} < \infty.$$

It is possible to generalize this theorem by rejecting the condition of continuity of functions.

Theorem 2. (see [1, 3]) *Let $0 \leqslant \alpha < 1$ and $1 \leqslant p < 1 + \alpha$. Let $f \in C^{1+\alpha}$, and let $h \in \mathcal{W}_p([0,T])$ be right-continuous. Then the equation*

$$y_t = a + (RRS) \int_0^t f(y_{s-})\, dh_s, \qquad 0 \leqslant t \leqslant T,$$

has the unique solution in $\mathcal{W}_p([0,T])$.

The existence of the solution to this equation is proved by using the Picard iteration method.

At present the results of this type are generalized in two directions. The first way is the consideration of integral equations by using more general integrals for discontinuous integrators (see [3]). The other way is to extend such integral equations for $p > 2$ (see [14]).

Now let us return back to the fBm with $1/2 < H < 1$. Since the fBm has locally a bounded p-variation for $p > 1/H$, the stochastic version of equation (0.1)

$$X_t = \xi + \int_0^t f(X_s)\, ds + (RS) \int_0^t g(X_s)\, dB_s^H, \qquad 0 \leqslant t \leqslant T$$

can be solved path-wise. If f is a Lipschitz continuous function and $g \in C^{1+\alpha}(\mathbb{R})$ then this equation has a unique solution in $C\mathcal{W}_p([0,T])$ (see [7, 16], also [6, 12]).

Clearly, the fBm is not a unique element of the class of processes having a locally bounded p-variation, $1 < p < 2$. Let $1 < p < \infty$. As an example of the process of locally bounded p-variation, one can take a zero-mean separable Gaussian stochastic process A with $v_{p,2}(\sigma_A; [0,T]) < \infty$ for each $0 < T < \infty$ (see [3]), where $\sigma_A(s,t) := (\mathbf{E}[(A(s) - A(t))^2])^{1/2}$,

$$v_{p,2}(\sigma_A; [0,T]) := \sup\left\{ \sum_{i=1}^n \varPsi_{p,2}(\sigma_A(t_{i-1}, t_i))\colon\ \{t_i\}_{i=0}^n \subset [0,T] \right\},$$

$$\varPsi_{p,2}(u) := \begin{cases} [u(\log\log(1/u))^{1/2}]^p, & \text{for } 0 < u \leqslant e^{-e}, \\ u^p, & \text{for } u > e^{-e}, \\ 0, & \text{for } u = 0. \end{cases}$$

Considering the Internet traffic models as a limit process (see [5]), it is possible to obtain a zero-mean, non-Gaussian and non-stable process $Y_\beta(t)$ with stationary increments and the covariance function

$$\sigma_\beta(t^{2-\beta} + s^{2-\beta} - |t-s|^{2-\beta}), \quad 0 < \beta < 1.$$

The trajectories of the process $Y_\beta(t)$ are Hölder continuous of order γ, for any $0 < \gamma < 1$.

Thus, for all these and other processes with p-bounded variation, $1 < p < 2$, one can construct and solve integral equations. The Gaussian property is not useful for these processes.

In the stochastic analysis there is a well-developed theory of stochastic differential equations driven by semimartingales, i.e., by processes that can be represented in the form $M + A$, where M is a local martingale and A is a process of locally bounded variation. As an example of a local martingale one can take Brownian motion.

A semimartingale is of bounded q-variation on any bounded interval for each $q > 2$. Thus, it is very natural to take a new class of processes, more general than semimartingales, into consideration.

Definition 1. (see [15]) For $p \in [1,2)$, an $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$ adapted cad-lag stochastic process Y is called a p-semimartingale if there exist stochastic processes M and A such that

$$Y - Y(0) = M + A \qquad \text{almost surely},$$

where $M(0) = A(0) = 0$, M is an $\mathbb{F}$ local martingale and A is an $\mathbb{F}$-adapted process with a locally bounded p-variation, i.e., for any fixed $T > 0$, the process $A = \{A_t, 0 \leqslant t \leqslant T\}$ has a bounded p-variation.

Consider the stochastic integral equation (SIE)

$$X_t = \xi + (SI)\int_0^t f(X_s)\,dM_s + (RS)\int_0^t g(X_s)\,dA_s, \qquad 0 \leqslant t \leqslant T. \quad (0.2)$$

By M we denote a local continuous martingale, and by A a continuous process with a locally bounded p-variation. The symbol SI denotes the usual stochastic integral.

Since almost all sample paths of a martingale have q-bounded variation, $q > 2$, we shall seek a solution of this equation in the class of processes whose almost all sample paths are in $C\mathcal{W}_q([0,T])$.

The existence of the solution of this equation is proved by using the Euler-Peano approximation

$$X_t^n = \xi + (SI)\int_0^t f(X_{s-}^{n,\varkappa^n})\,dM_s + (RS)\int_0^t g(X_{s-}^{n,\varkappa^n})\,dA_s, \quad t \geqslant 0,\ n \in \mathbb{N},$$

where $X_t^{n,\varkappa^n} = X(t_k^n)$ for $t \in [t_k^n, t_{k+1}^n)$; $\varkappa^n = \{t_k^n : k \geqslant 0\}$ is a sequence of partitions of $[0,\infty)$, i.e., $0 = t_0^n < t_1^n < t_2^n < \cdots$, $\lim_{k\to\infty} t_k^n = \infty$, such that for every $T > 0$ we have

$$\max_{k \le r^n(T)} |t^n_{k+1} - t^n_k| \to 0, \ \ n \to +\infty,$$

where $r^n(T) = \max\{k : t^n_k \le T\}$.

Theorem 3. (see [8], [10]) *Let $0 < \alpha < 1$, $q > 2$, and $1 < p < 2$ be such that $\frac{\alpha}{q} + \frac{1}{p} > 1$. Let f be a Lipschitz continuous function, and let $g \in C^{1+\alpha}(\mathbb{R})$. Then there exists a unique strong solution of equation (0.2).*

Corollary 1. The Euler-Peano approximation X^n converges in probability to the strong solution of equation (0.2).

Remark 2. It is possible to generalize this theorem to quasi-left continuous processes M and A.

As it has been stated above, the SIE

$$X_t = \xi + (SI) \int_0^t f(X_s)\, dW_s + (RS) \int_0^t g(X_s)\, dB^H_s, \qquad 0 \leqslant t \leqslant T \quad (0.3)$$

can be approximated by the Euler-Peano scheme. But if we are interested in the convergence rate, then we need an intermediate scheme between Euler-Peano and Milstein. For each $n \geqslant 1$, we define the approximation

$$\begin{aligned} X^n(t) = \xi &+ \int_0^t f(X^{n,\varkappa^n}_{s-})\, dW_s + \int_0^t g(X^{n,\varkappa^n}_{s-})\, dB^H_s \\ &+ \int_0^t \int_{\rho^n_s}^s g'(X^{n,\varkappa^n}_{s-}) f(X^{n,\varkappa^n}_{s-})\, dW_u\, dB^H_s \\ &+ \int_0^t \int_{\rho^n_s}^s g'(X^{n,\varkappa^n}_{s-}) g(X^{n,\varkappa^n}_{s-})\, dB^H_u\, dB^H_s, \end{aligned}$$

where $\rho^n(s) = \max\{t^n_k\colon\ t^n_k \leqslant s\}$.

Theorem 4. [9] *Let f and g be bounded functions, f be a Lipschitz function and $g \in C^2(\mathbb{R})$. Then*

$$\alpha_n V_q\big(X^n - X; [0,T]\big) \xrightarrow{\mathbf{P}} 0, \qquad n \to \infty,$$

where $\delta_n = \max_k |t^n_k - t^n_{k-1}|$, $\alpha_n = \delta_n^{-1/q} |\ln \delta_n|^{-1/2}$, $\delta_n < 1$, $q > 2$.

Assume that $H > 1/2$. In this case we have the following kernel representation of B^H with respect to the standard Brownian motion

$$B^H_t = \int_0^t K_H(t,s)\, \mathrm{d}W_s$$

with a deterministic kernel

$$K_H(t,s) = c_H s^{-\alpha} \int_s^t u^\alpha (u-s)^{\alpha-1}\, \mathrm{d}u,$$

where $\alpha = H - 1/2$, c_H is the normalizing constant.

Let $\varkappa^n = \{t_k^n\colon\ 0 \leqslant k \leqslant n\}$ be a sequence of partitions of the interval $[0,1]$ such that $t_k^n = k/n$ and let

$$A_t^n = \sum_{k=1}^{[nt]} n \int_{t_{k-1}^n}^{t_k^n} K_H\Big(\frac{[nt]}{n}, u\Big) du \cdot \big(W(t_k^n) - W(t_{k-1}^n)\big)\,, \qquad M_t^n = W_t^{\varkappa^n},$$

where $[nt]$ denotes the integer part of nt. Define

$$X_t^n = \xi + \int_0^t f(X_{s-}^n)\,\mathrm{d}M_s^n + \int_0^t g(X_{s-}^n)\,\mathrm{d}A_s^n, \qquad 0 \leqslant t \leqslant 1.$$

Theorem 5. [11] *Let* $0 < \alpha < 1$, $q > 2$, *and* $1 < p < 2$ *be such that* $\frac{\alpha}{q} + \frac{1}{p} > 1$. *Let* f *be a bounded Lipschitz-continuous function, and let* g *be bounded,* $g \in C^{1+\alpha}(\mathbb{R})$. *Assume* $m \in \{0, 2, 4, \ldots, 2^k, \ldots\}$. *Then*

$$\sup_{t \leqslant 1} |X_t^m - X_t| \xrightarrow{\mathbf{P}} 0,$$

where X *is the solution of equation* (0.3).

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