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APPLICATIONS OF THE METHOD OF STATIONARY PHASE TO ASYMPTOTIC INTEGRATION OF WEAKLY NONLINEAR HYPERBOLIC SYSTEMS

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Abstract. Hyperbolic weakly nonlinear system with periodical initial conditions is considered in the article. The unperturbated system describes non interacted travelling waves. Perturbated system describes some interaction of the waves, which can be complicated by resonances. The asymptotic solution of the problem can be finds as a solution of system, averaging along characteristics. The characteristics of system depend from slowly time and the problem of asymptotic substantiation is more difficult compare with constant characteristics case. This substantiation was makes in early work of the author for one class of solutions. In this article other class of solutions is considered and the other technics (method of stationary phase) is used to substantiate the asymptotic.

Key words: nonliniear waves, hyperbolic systems, perturbations methods, averaging, resonaces

1. Statment of Problem

We consider a hyperbolic system of weakly nonlinear differential equations with a small positive parameter ε:

$$\frac{\partial u_j}{\partial t} + \lambda_j(\varepsilon t)\frac{\partial u_j}{\partial x} = \varepsilon f_j(u_1, \dots, u_n) \tag{1.1}$$

and with periodic initial contitions:

$$u_j(0, x) = u_{0j}(x) \equiv u_{0j}(x + 2\pi). \tag{1.2}$$

In [6] the method of asymptotic integration of the (1.1), (1.2) system was presented. This method gives the asymptotic solution

$$u_j = v_j(\tau, y_j) + O(\varepsilon),\ \ \tau = \varepsilon t, \qquad (1.3)$$
$$y_j = x - \frac{1}{\varepsilon}\int_0^{\tau} \lambda_j(\tau)\, d\tau,$$

which is uniformly valid in the domain $t \in \left[0, O(\varepsilon^{-1})\right]$. The asymptotic solution $V = (v_1, \dots, v_n)$ was sought as the solution of the averaging system

$$\frac{\partial v_j}{\partial \tau} = \langle f_j(V) \rangle_j\,,\ \ v_j(0, y_j) = u_{0j}(y_j). \qquad (1.4)$$

The functions $f_j(V)$ were averaged along the characteristics of the nonperturbated system

$$\frac{\partial u_j}{\partial t} + \lambda_j(0)\frac{\partial u_j}{\partial x} = 0.$$

This scheme of the averaging of system (1.1) was used in [1] under strong requirement for the coefficients $\lambda_j(\tau)$:

$$\frac{d}{d\tau}\left(\frac{\lambda_i(\tau) - \lambda_j(\tau)}{\lambda_k(\tau) - \lambda_j(\tau)}\right) \equiv 0,\ \ \forall i, j, k. \qquad (1.5)$$

Therefore the method from [1] can be used only if all coefficients λ_j are some combinations of one function $\alpha(\tau)$: $\lambda_j(\tau) = \lambda_j^0 \alpha(\tau) + \lambda_0$.

In this work the coefficients $\lambda_j(\tau) \in C^1\left[0, \tau_0\right]$ of systems (1.1) satisfy the following requirements: with all indexes j, s, p and $\forall \tau \in [0, \tau_0]$:

$$\lambda_s(\tau) \neq \lambda_p(\tau), \qquad (1.6)$$
$$W(\tau) \equiv \begin{vmatrix} 1 & 1 & 1 \\ \lambda_j(\tau) & \lambda_s(\tau) & \lambda_p(\tau) \\ \frac{d\,\lambda_j(\tau)}{d\tau} & \frac{d\,\lambda_s(\tau)}{d\tau} & \frac{d\,\lambda_p(\tau)}{d\tau} \end{vmatrix} \neq 0.$$

Conditions (1.5) are not compatible with (1.6) conditions and therefore in this work we consider the new class of (1.1), (1.2) problems.

2. Method of Asymptotic Integration

We will consider the following functions f_j in (1.1):

$$f_j(u_1, \dots, u_n) = \sum_{s=1}^{n}\sum_{p=1}^{n} f_{jsp}(u_s, u_p). \qquad (2.1)$$

This type of f_j represents the typical problems of the asymptotic integrations of (1.1),(1.2) system. Note that applications of (1.1) models has quadratic nonlinearities [4, 5].

Asymptotic solution of (1.1), (1.2), (2.1) problem is the solution of the integro–differential system:

$$\frac{\partial v_j}{\partial \tau} = f_{jjj}(v_j, v_j) + \frac{1}{2\pi}\sum_{s\neq j}\int_0^{2\pi} f_{jsj}(v_s(\tau, y), v_j)\,dy$$
$$+ \frac{1}{2\pi}\sum_{p\neq j}\int_0^{2\pi} f_{jjp}(v_j, v_p(\tau, z))\,dz \tag{2.2}$$
$$+ \frac{1}{4\pi^2}\sum_{s\neq j}\sum_{p\neq j}\int_0^{2\pi}\int_0^{2\pi} f_{jsp}(v_s(\tau, y), v_p(\tau, z))\,dy\,dz,$$
$$v_j(0, y_j) = u_{0j}(y_j).$$

Let f_{jsp} and u_{0j} be continuously differentiable functions. Then system (2.2) has only one periodic solution

$$V = (v_1, \dots, v_n), v_j(\tau, y_j) \equiv v_j(\tau, y_j + 2\pi), \quad \tau \in [0, \tau_0'] \,.$$

Constant $\tau_0' \leq \tau_0$ and all the other constants care independent from ε. Exact solution $(u_1, \dots, u_n) = U(t, x; \varepsilon)$ of the (1.1),(1.2) system exists if $t \in \left[0, \frac{\tau_0''}{\varepsilon}\right]$. Constant $\tau_0'' \leq \tau_0$, where $\tau_0 = \min\{\tau_0', \tau_0''\}$ is denoted by τ_0.

3. Substantiation of the Averaging

The difference of the exact and asymptotic solution is given by:

$$r_j(t, x; \varepsilon) = u_j(t, x; \varepsilon) - v(\varepsilon t, x - \frac{1}{\varepsilon}\int_0^{\varepsilon t} \lambda_j(\tau)\,d\tau).$$

Functions r_j must satisfy

$$\frac{\partial r_j}{\partial t} + \lambda_j(\varepsilon t)\frac{\partial r_j}{\partial x} = \varepsilon\left(\sum_{k=1}^{n} h_{jk}(t, x; \varepsilon) r_k + \mu_j(t, x; \varepsilon)\right) \tag{3.1}$$

with zero initial conditions:

$$r_j(0, x; \varepsilon) = u_j(0, x; \varepsilon) - v_j(0, x) = u_{0j}(x) - u_{0j}(x) \equiv 0. \tag{3.2}$$

The functions $\mu_j(t, x; \varepsilon)$ in (3.1) are given by

$$\mu_j = \left(f_j(\cdots) - \frac{\partial v_j}{\partial \tau}\right)\Bigg|_{\substack{\tau = \varepsilon t \\ y_k = x - \frac{1}{\varepsilon}\int_0^{\varepsilon t}\lambda_k(\tau)\,d\tau}} . \tag{3.3}$$

All functions $v_k(\tau, y_k)$ in (3.3) are periodic. Therefore the μ_j can be writen as Fourier series:

$$\mu_j \sim \sum_{l_s, l_p \in Z} \mu_{j l_p l_s}(\tau) \exp\{i l_s y_s + i l_p y_p\}. \tag{3.4}$$

We demand, that functions f_{jsp} and u_{0j} be smooth and the series covergent:

$$\sum_{l_s,l_p\in Z}(|l_s|+|l_p|)\left(\left|\mu_{jl_pl_s}(\tau)\right|+\left|\frac{d}{d\tau}\mu_{jl_pl_s}(\tau)\right|\right).$$

We integrate (3.1),(3.2) system along characteristics $x-\frac{1}{\varepsilon}\int_0^{\varepsilon t}\lambda_j(\tau)\,d\tau = const$:

$$r_j(t,x;\varepsilon)=\varepsilon\int_0^t\left(\sum_{k=1}^n h_{jk}r_k+\mu_j(\tilde{t},x-\frac{1}{\varepsilon}\int_{\varepsilon\tilde{t}}^{\varepsilon t}\lambda_j(\tau)\,d\tau;\varepsilon)\right)d\tilde{t}\,. \quad (3.5)$$

We substitute formula (3.4) into system (3.5) and give the integrals of harmonics:

$$\varepsilon\int_0^t\mu_{jl_sl_p}(\varepsilon\tilde{t})\exp(il_s(x-\frac{1}{\varepsilon}\int_0^{\varepsilon t}\lambda_j(\tau)\,d\tau+\frac{1}{\varepsilon}\int_0^{\varepsilon\tilde{t}}(\lambda_j(\tau)-\lambda_s(\tau))\,d\tau)$$
$$+il_p(x-\frac{1}{\varepsilon}\int_0^{\varepsilon t}\lambda_j(\tau)\,d\tau+\frac{1}{\varepsilon}\int_0^{\varepsilon\tilde{t}}(\lambda_j(\tau)-\lambda_p(\tau))\,d\tau))\,d\tilde{t}$$
$$=\exp(i(l_s+l_p)y_j)\int_0^\tau\mu_{jl_sl_p}(\tilde{\tau})\exp(i\frac{1}{\varepsilon}\delta_{jl_sl_p}(\tilde{\tau}))\,d\tilde{\tau}.$$

There

$$\delta_{jl_sl_p}(\tau)=\int_0^\tau(l_s(\lambda_j(\tau)-\lambda_s(\tau))+(l_p(\lambda_j(\tau)-\lambda_p(\tau)))\,d\tau.$$

Note, that the averaging system (2.2) has no members, such that

$$\delta_{jl_sl_p}(\tau)\equiv 0.$$

If the l_s and l_p are integer numbers, such that

$$\left|\delta'_{jl_sl_p}(\tau)\right|>\nu(\varepsilon). \quad (3.6)$$

Then we integrate the inequality (3.6):

$$\left|\int_0^\tau\mu_{jl_sl_p}(\tau)\exp\left(\frac{i\delta_{jl_sl_p}(\tau)}{\varepsilon}\right)d\tau\right|$$
$$<\frac{\varepsilon}{\nu(\varepsilon)}\left(\left\|\mu_{jl_sl_p}\right\|+\left\|\mu'_{jl_sl_p}\right\|(|l_s|+|l_p|)\frac{\varepsilon}{\nu(\varepsilon)}\Lambda_1\right).$$

Here $\|\cdot\|$ is the maximum of the functions in the interval $\tau\in[0,\tau_0]$, constant $\Lambda_1\geq\left\|\lambda'_j(\tau)-\lambda'_k(\tau)\right\|$, H_ε is the set of all nonresonanace harmonics, c_0 is a positive constant:

$$\left|\sum_{l_s,l_p\in H_\varepsilon}\exp\left(i(l_s+l_p)y_j\right)\int_0^\tau\mu_{jl_sl_p}(\tau)\exp\left(\frac{i\delta_{jl_sl_p}(\tau)}{\varepsilon}\right)d\tau\right|\leq c_0\frac{\varepsilon}{\nu(\varepsilon)}.$$

If condition (3.6) is not valid, i.e. $l_s, l_p \notin H_\varepsilon$, we have a resonance case. Denote $\delta'_{jl_sl_p} = \alpha$, $\delta''_{jl_sl_p} = \beta$ and consider the linear system

$$\begin{cases} l_s(\lambda_j(\tau) - \lambda_s(\tau)) + l_p(\lambda_j(\tau) - \lambda_p(\tau)) = \alpha, \\ l_s(\lambda'_j(\tau) - \lambda'_s(\tau)) + l_p(\lambda'_j(\tau) - \lambda'_p(\tau)) = \beta. \end{cases} \tag{3.7}$$

It follows from (1.6) that the determinant of (3.7)

$$\begin{vmatrix} \lambda_j - \lambda_s & \lambda_j - \lambda_p \\ \lambda'_j - \lambda'_s & \lambda'_j - \lambda'_p \end{vmatrix} \equiv W(\tau) \neq 0, \ \forall \tau \in [0, \tau_0] \,.$$

Let point τ' be local extreme of $\delta'_{jl_sl_p}(\tau)$. Therefore, in this point system's (3.7) coefficient $\beta = 0$ and $|\alpha| > W_0/\Lambda_1$, constant $W_0 = \min\limits_{\tau \in [0,\tau_0]} |W(\tau)| > 0$.

Then (3.6) unavailable only in the around the point s ($\delta'_{jl_sl_p}(s) = 0$). Therefore, in system (3.7) $\alpha = 0$ and

$$|\beta| > W_0/\Lambda_0 = \beta_0, \quad \Lambda_0 \geq \|\lambda_j(\tau) - \lambda_k(\tau)\| \,.$$

If $\alpha = 0$, then nonzero solution of system (3.7) ($|l_s| + |l_p| \neq 0$) is given by

$$\varphi(\tau) \equiv \frac{\lambda_j - \lambda_s}{\lambda_j - \lambda_p} = -\frac{l_p}{l_s}.$$

Function $\varphi(\tau)$ is monotonic:

$$\varphi'(\tau) = \frac{W(\tau)}{(\lambda_j - \lambda_p)^2} \neq 0.$$

Therefore, function $\delta'_{jl_sl_p}(\tau)$ can have only one stationary point $s \in (0, \tau_0)$. Let us use the method of stationary phase and consider three intervals:

$$[0, \, s - \eta(\varepsilon)] \,, \ \ (s - \eta(\varepsilon), \, s + \eta(\varepsilon)) \,, [s + \eta(\varepsilon), \, \tau_0] \,.$$

In the first and third intervals: $\left|\delta'_{jl_sl_p}(\tau)\right| > \beta_0 \eta(\varepsilon)$. Therefore there exist positive constants c_1 and c_2:

$$\left| \sum_{l_s, l_p \notin H_\varepsilon} \exp(i(l_s + l_p)y_j) \int_0^\tau \mu_{jl_sl_p}(\tau) \exp(\frac{i\delta_{jl_sl_p}(\tau)}{\varepsilon}) \, d\tau \right| \\ \leq c_1 \frac{\varepsilon}{\eta(\varepsilon)} + c_2 \eta(\varepsilon). \tag{3.8}$$

The function $\nu(\varepsilon)$ in (3.6) can be replaced by constant W_0/Λ_1. (3.8) is valid with any function $0 < \eta(\varepsilon) < \tau/2$. Therefore, if $\eta(\varepsilon) = \sqrt{\varepsilon}$, we have

$$r_j(t, x; \varepsilon) = \varepsilon \left(\sum_{k=1}^{n} \int_0^t h_{jk} r_k \, d\tilde{t} \right) + O(\sqrt{\varepsilon}).$$

Therefore,

$$|r_j(t, x; \varepsilon)| < c_3 \sqrt{\varepsilon}, \forall t \in \left[0, \frac{\tau^0}{\varepsilon}\right], \ x \in [0, 2\pi] \,.$$

4. Conclusion

The main result of this work is the following: For system (1.1), (1.2), (2.1) under conditions (1.6) averaging system (2.2) defines the following asymptotic

$$u_j = v_j(\tau, y_j) + O(\sqrt{\varepsilon}), \ \ \tau = \varepsilon t, \ \ y_j = x - \frac{1}{\varepsilon}\int_0^\tau \lambda_j(\tau)\, d\tau.$$

which is uniformly valid in the domain $t \in \big[0, O(\varepsilon^{-1})\big]$.

Note, that the (4.1) asymptotic solution is only of the $O(\varepsilon)$ order approximation of the exact solution. It is not a limitation of the method, but it is a price we should pay for more difficult class of conditions (1.6), compared with conditions (1.5). In particular, if the problem satisfies nonresonance conditions, we have $\nu = const$ in the inequality (3.6). Therefore, do not exist harmonics l_s, l_p, which supply the conditions (3.8) and the order of the asymptotic approximation in this case is $O(\varepsilon)$.

The (1.1) system generalizes the case of constant coefficients λ_j, which appears as the problem of long waves asymptotic write in Riemann invariants [4, 5]. Similar problems were considered in [1, 7]. A survey of mathematical results in asymptotic methods for waves interactions was presented in [2], see also [3].

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