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SEGMENTATION COMBINING APPROACHES BASED ON MEAN CURVATURE

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Abstract. This paper presents a discussion on problem of segmentation using the level set method and the mean curvature problem for graphs. Its contribution consists in combining the existing equations via the regularization parameter. To obtain a numerical solution we use the finite volume method.

Key words: segmentation, level set equation, mean curvature flow for graphs, finite volume method

1. Level Set Equation

Suppose we are given an interface separating one region from another. We are given a speed k that tells how to move each point of the interface. The *level set* approach, introduced by Osher and Sethian takes the original interface – e.g. a curve in two dimensions – and embeds it into a surface in such way, that the interface represents a level line of the surface called *level set surface*. The idea is that instead of moving the interface, we move the corresponding level set surface. To find where the interface is, we cut the surface at the level corresponding to the interface.

Let $\overline{x}(t)$ be the moving curve of the level set function u, i.e. $u(\overline{x}(t), t) = c$, then differentiating in time we have

$$\partial_t u = -\nabla u . \partial_t \overline{x}. \tag{1.1}$$

The speed k of the level line will depend on its (mean) curvature, given by $k = \nabla . \left(\frac{\nabla u}{|\nabla u|}\right)$ and we move the level lines in the direction opposite to the outward normal, given by $\frac{\nabla u}{|\nabla u|}$. Thus

$$\partial_t \overline{x} = -k\vec{N} = -k\frac{\nabla u}{|\nabla u|}.$$

After substituting it into (1.1) and rearranging the terms we get

$$\partial_t u = |\nabla u| \nabla . \left(\frac{\nabla u}{|\nabla u|} \right) . \tag{1.2}$$

The equation (1.2) has only the viscosity solution, thus to be able to speak about a variational solution, the equation must be regularized. We regularize the equation (1.2) in the sense of Evans and Spruck [2], where

$$|\nabla u| \approx |\nabla u|_\varepsilon = \sqrt{\varepsilon^2 + |\nabla u|^2}.$$

To finish the formulation of the level set problem we have

$$\frac{\partial_t u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} - \nabla . \left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right) = 0 \quad \text{in } Q_T \equiv I \times \Omega, \tag{1.3}$$

$$u(x,0) = u^0(x) \ \text{ in } \Omega,$$

$$u(x,t) = u^0(x) \quad \text{on } I \times \partial\Omega,$$

where u is a computed level set function, $\Omega \subset I\!R^d$ is a rectangular computational domain, $\varepsilon > 0$ is a parameter, $I = (0, T]$ is a time interval.

2. Level Set Equation in Segmentation

Segmentation can be defined as a piecewise constant graph that varies rapidly across the boundaries between different objects and stays flat within it.

Now we will evolve an initial surface with corresponding function called *segmentation function*. The segmentation will be the piecewise constant approximation of this surface, and not the approximation of the image itself. For the purpose of segmentation, the equation (1.3) can be modified in such way, that we modulate the speed of the level line by $g(|\nabla I_0|)$, where I_0 is the segmented image and for a function g we take the function $g(s) = 1/(1+Ks^2)$. The speed of the level line is slower in areas with large gradients usually typical for edges and faster in areas with small gradients usually typical for "inner areas" and for noise. We get the equation

$$\partial_t u = g(|\nabla I_0|) |\nabla u|_\varepsilon \nabla . \left(\frac{\nabla u}{|\nabla u|_\varepsilon} \right) . \tag{2.1}$$

We have now two inputs for this equation: one is the segmented image I_0 and the other is the segmentation function u, where u_0 is a "peak" function obtained by $u_0(x,y) = K/(1 + \sqrt{(x - s_x)^2 + (y - s_y)^2})$, where s_x and s_y are the coordinates of the "peak". K is the scale, and is usually set in dependence to the discretization.

During the evolution all level lines of the function u shrink with the speed depending on their curvature, except of the level lines in the vicinity of the image edges, where, due to the antradienis, spalis 11, 2005 at 5:00 pmfunction

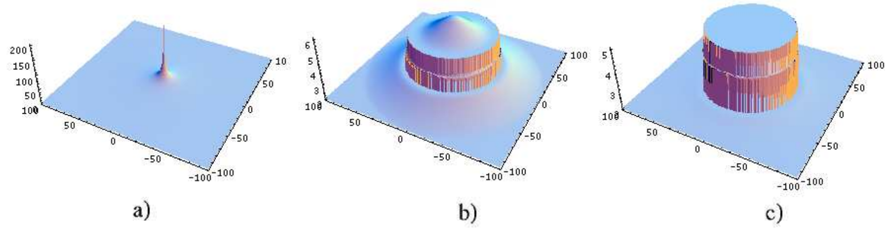

Figure 1. The segmentation according to (2.1).

g, the speed is slowed down. The "steady state" of a particular level line corresponds to a boundary of a segmented object. An example of segmentation is depicted in Fig. 1.

If the boundary of a segmented object is not closed, the algorithm is still able to detect the object and complete the missing part of the boundary with a line segment. After spilling out of the segmentation function across the boundary, its level lines around the missing part of the boundary become curved and principles of the curvature driven motion are applied.

The curve evolution and the level set models for segmentation have been significantly improved by finding a proper driving force in the form $-\nabla g(|\nabla I_0(x)|)$ [1, 3]. The vector field $-\nabla g(|\nabla I_0(x)|)$ has the important geometric property: *it points towards regions where the norm of the gradient* ∇I_0 *is large.* Thus if an initial curve belongs to a neighborhood of an edge, then it is driven towards this edge by this velocity field. We get the equation

$$\partial_t u = |\nabla u|_\varepsilon \nabla . \left(g(|\nabla I_0|) \frac{\nabla u}{|\nabla u|_\varepsilon} \right). \tag{2.2}$$

3. Mean Curvature Problem for Graphs

The parameter ε can be used not only as a regularization factor, but also as a modeling parameter. Because of ε, the level set form of the mean curvature flow is closely related to the mean curvature problem for graphs. If we have a graph $\Gamma(t) = \{(x, u(x,t)) | x \in \Omega\}$ and normal to Γ is chosen to be $\nu(u) = \frac{(\nabla u, -1)}{Q(u)}$, $Q(u) = \sqrt{1 + |\nabla u|^2}$, then the normal velocity to Γ can be written as $V(u) = -\frac{u_t}{Q(u)}$. The mean curvature of Γ is given by $H(u) = \nabla . \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$. Equation $V = -H$, where V represents the normal velocity and H is the mean curvature of the graph leads to a differential equation

$$\frac{\partial_t u}{\sqrt{1 + |\nabla u|^2}} = \nabla . \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \tag{3.1}$$

which is equal to (1.3) for $\varepsilon = 1$. In Fig. 2b) and Fig. 2c) we can see the result of processing the original data from the (Fig. 2a)) by the finite volume scheme based on (3.1) after 5 resp. 50 time steps. The result of processing by the level set algorithm is visually very close to Fig. 2a): there is almost no change because of the zero curvature in the normal direction to the level sets.

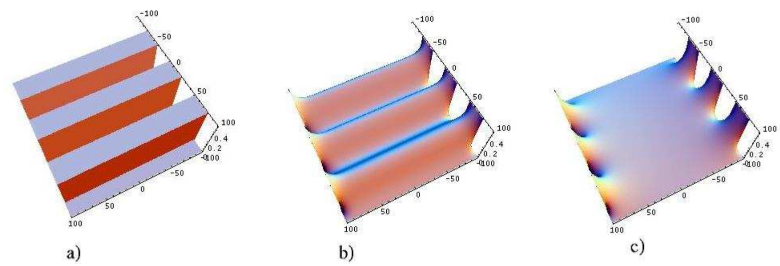

Figure 2. ε as a modeling factor.

4. Combining the Level Set Equation and Mean Curvature for Graphs

(1.3) represents the mean curvature problem for graphs for $\varepsilon = 1$ and level set equation for ε close to zero. Similarly, the equation (2.2) presents a segmentation problem based on the mean curvature flow for graphs for $\varepsilon = 1$ and on the level set equation for ε close to zero.

Let us start segmentation with $\varepsilon = 10^{-6}$ for a certain number of steps necessary to form shocks on the part of the missing boundary (Fig. 3a)). Then, to make the segmentation function flat, we use model

$$\frac{\partial_t u}{\sqrt{(g_2(|\nabla u|))^2 + |\nabla u|^2}} = \nabla . \left(\frac{g(|\nabla I_0|)\nabla u}{\sqrt{(g_2(|\nabla u|))^2 + |\nabla u|^2}} \right), \tag{4.1}$$

where g_2 is the following function:

$$g_2(v) = \begin{cases} 10^{-6}, & \text{for } 0 \leq t \leq T_1, \\ \dfrac{1}{1 + Kv^2}, & \text{for } T_1 < t \leq T. \end{cases}$$

Fig. 3 displays the following experiment. The image on the left depicts the result at T_1 ($T_1 = 40$), when shocks are developed. Generally, T_1 can be set to the time, in which the segmentation function stops to change. Then ε in the algorithm changes according to the gradients: this parameter stays small on the highest gradients where shocks have to be kept and is greater for smaller gradients, where we wish to flatten the segmentation function. Images b) and c) show the evolution in the middle and final steps.

Though the parabolic equation (4.1) is nonlinear, a theory in [4] can be applied to show that it has a unique weak solution in $L_2(I, H^1(\Omega))$.

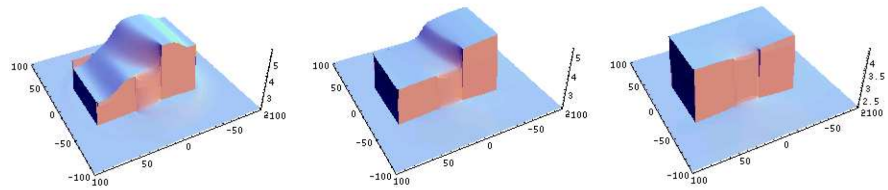

Figure 3. Model (4.1). a) forming the shock, b) and c) flattening of the segmentation function.

Deriving the finite volume scheme. To obtain a numerical solution we use a semi-implicit scheme based on the finite volume method. Its advantage is, that it is stable for any size of time step parameter. We will subdivide the continuous computational domain into rectangular regions and look for a solution, which will be constant over each such region - *control volume*.

First we integrate equation over a control volume $p \subset \Omega$. After using Green's theorem, for any p we get the integral identity

$$\int\limits_{p} \frac{\partial_t u}{\sqrt{(g_2(|\nabla u|))^2 + |\nabla u|^2}} dx - \int\limits_{\partial p} \frac{g(|\nabla I_0|)\nabla u}{\sqrt{(g_2(|\nabla u|))^2 + |\nabla u|^2}} . \vec{\nu}\ ds = 0, \tag{4.2}$$

where $\vec{\nu}$ is the outward unit normal vector to the boundary ∂p of the control volume p. To discretize the integral form (4.2) in time we choose N as the number of time steps and obtain the length of a uniform discrete time step $k = \frac{T}{N}$. Then at any discrete time $t_n = nk, n = 1, \ldots, N$ we replace the time derivative by the backward difference, i.e. $\partial_t u$ by $\dfrac{u^n - u^{n-1}}{k}$, where u^n, u^{n-1} are solutions of (4.2) at times $t_n = nk$, $t_{n-1} = (n-1)k$, respectively.

We treat the spatial nonlinear terms of the equation using solution from the previous time step and use approximation of the linear terms at the current time level. Let value of u^n over p be denoted by $\overline{u}_p^n$ and the value of a mirror extension of u^n - $\tilde{u}^n$ over p be denoted by $\tilde{u}_p^n$. Let τ be a mesh of Ω with cells (control volumes) p of measure $m(p)$. For every cell p we consider a set of neighbors $N(p)$ consisting of all cells $q \in \tau_h$ for which common interface of p and q is a line segment e_{pq} of non-zero measure $m(e_{pq})$. We assume that for every p, there exists a representative point $x_p \in p$ such, that for every pair $p, q, q \in N(p)$, the vector $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to unit vector $\vec{\nu}_{pq}$ which is normal to the common interface e_{pq} and oriented from p to q. Here, x_p is set just as a center of the pixel. Let x_{pq} be the intersection of the line segment e_{pq} and the segment $\overline{x_p x_q}$. Then we define *transmitivity coefficients* $T_{pq} := \frac{m(e_{pq})}{|x_q - x_p|}$.

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_{N_{\max}} = T$ denote the time discretization with $t_n = t_{n-1} + k$, where k is the time step. For $n = 0, \ldots, N_{\max} - 1$ we look for $\overline{u}_p^{n+1}$, $p \in \tau$ satisfying

$$\frac{\overline{u}_p^{n+1}-\overline{u}_p^{n}}{k|\nabla \tilde{u}_p^n|}\, m\,(p) = \sum_{q\in N(p)} T_{pq} g(|\nabla I_0|)\, \frac{\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right)}{\sqrt{(g_2(|\nabla \tilde{u}_{pq}^n|))^2 + |\nabla \tilde{u}_{pq}^n|^2}}. \quad (4.3)$$

The system matrix is a sparse M-matrix, so the linear system is uniquely solvable. It can be solved e.g. by the Gauss-Seidel method, eventually improved by the SOR method. In the scheme we work with two types of gradients: $\nabla \tilde{u}_{pq}^n$ computed on the control volume boundary ∂p and $\nabla \tilde{u}_p^n$ computed on the control volume p. Let us repeat, that $\tilde{u}^n$ is now mirror extension of the data computed in the n-th discrete time step and $\tilde{u}_p^n$, $\tilde{u}_{pq}^n$ denotes its approximate values inside p and along ∂p, respectively.

Remark 1. (Computing the gradients on ∂p). The gradients of u along the boundary ∂u are approximated by their values in x_{pq}, $q \in N(p)$. In 2D, we compute the gradients approximately with the stencil given by Fig. 4. For horizontal boundary segments, the stencil is rotated. The stencil corresponds to computing $|\nabla G_\sigma * u|$ instead of $|\nabla u|$ using suitable kernel G_σ. Let us note, that $|\nabla G_\sigma * u| < C_\sigma$.

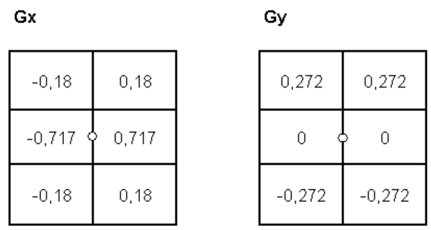

Figure 4. The small circles denote the position of x_{pq}.

Remark 2. (Computing the gradients on a control volume p). We use simple averaging given (in case of regular square grids) by

$$|\nabla \tilde{u}_p^n| \approx \frac{1}{\mathrm{card}(N(p))} \sum_{q\in N(p)} \sqrt{(g_2(|\nabla \tilde{u}_{pq}^n|))^2 + |\nabla \tilde{u}_{pq}^n|^2}. \quad (4.4)$$

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