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THE INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR PSEUDOPARABOLIC EQUATIONS

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Abstract. Various physical phenomena lead to a mixed boundary value problems or Cauchy problem for the partial differential equation $u_t - \Delta u - \eta \Delta u_t = 0$. Such equations are used to model fluid flow in fissured porous media, two phase flow in porous media with dynamical capillary pressure, heat conduction in two-temperature systems, flow of some non-Newtonian fluids and other [1, 2, 4, 5, 6]. The aim of this article is to show existence and uniqueness of weak solutions for some types of nonlinear pseudoparabolic equation.

Key words: pseudoparabolic equations, weak solution, initial-boundary value problem

1. Problem Formulation

Let Ω be a bounded domain in the Euclidean space R^n_x,

$$Q = \Omega \times (0;T), \quad T < \infty$$

is a cylindrical domain and $\mathrm{S} = \partial\Omega \times (0;T)$. In the domain Q we consider the following initial-boundary value problem for the nonlinear pseudoparabolic equation

$$\begin{cases} -\left(m^{ij}(u)u_{tx_j}\right)_{x_i} + m(u)u_t - \left(l^{ij}(x,t)u_{x_j}\right)_{x_i} + l(x,t)u \\ \qquad\qquad\qquad\qquad = f(x,t) + f^i_{x_i}(x,t), \\ u|_S = 0, \\ u|_{t=0} = \varphi(x). \end{cases} \tag{1.1}$$

The existence of a solution will be ensured by the following assumptions:

(A1) $f, f^i \in L_2(Q)$;

(A2) $\varphi \in H_0^1(\Omega)$;

(A3) $m^{ij}(r) : R \to R$ - continuous bounded functions, such that for every $\xi \in R_\xi^n$ and $r \in R$ $m_{00}|\xi|^2 \le m^{ij}(r)\xi_i\xi_j \le M|\xi|^2$, where m_{00} and M positive constants;

(A4) $m(r) : R \to R$ - continuous bounded function, such that $0 \le m(r) \le M$ for every $r \in R$;

(A5) $l^{ij}(x,t), l(x,t), l_t^{ij}(x,t), l_t(x,t)$ - measurable and bounded function on Q;

(A6) $l^{ij}(x,t) = l^{ji}(x,t)$ in Q;

(A7) $l^{ij}(x,t)\xi_i\xi_j \ge 0$ in Q for every $\xi \in R_\xi^n$;

(A8) $l(x,t) \ge 0$ in Q;

(A9) $\exists$ const $\mu \ge 0$, such that for every $\xi \in R_\xi^n$ in Q:

(a) $(\mu l^{ij} - \frac{1}{2} l_t^{ij})\xi_i\xi_j \ge 0$,

(b) $(\mu l - \frac{1}{2} l_t) \ge 0$.

2. Main Results

By $H_{1,1}(Q, S \bigcup \Omega)$ we denote the closure of the set of function such that $u(x,t) \in C^\infty(Q)$ and which vanish identically on neighborhood of $S \bigcup \Omega$, with respect to the norm

$$||u||_{H_{1,1}(Q)} = \Big(\int\limits_Q \big(u^2 + u_t^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i=1}^n u_{tx_i}^2\big)\, dxdt \Big)^{\frac{1}{2}}. \tag{2.1}$$

Similarly we define the space $H_{1,0}(Q,S)$ with norm

$$||u||_{H_{1,0}(Q)} = \Big(\int\limits_Q \big(u^2 + \sum_{i=1}^n u_{x_i}^2\big)\, dxdt \Big)^{\frac{1}{2}}. \tag{2.2}$$

Definition 1. We say that $u(x,t)$ is a weak solution of initial-boundary value problem (1.1), if $u(x,t) - \varphi(x) \in H_{1,1}(Q, S \bigcup \Omega)$ and for any $v(x,t) \in H_{1,0}(Q,S)$ the integral identity holds

$$\int\limits_Q \big(m^{ij}(u)u_{tx_j}v_{x_i} + m(u)u_t v + l^{ij}u_{x_j}v_{x_j} + luv\big)\, dxdt = \int\limits_Q (fv - f^i v_{x_i})\, dxdt. \tag{2.3}$$

Theorem 1. (Existence). *Suppose that assumptions* (A1) – (A9) *are satisfied. Then there exists a weak solution of the initial–boundary value problem* (1.1).

Proof. We first consider problem (1.1) with homogeneous initial condition $\varphi(x) = 0$ on Ω. We fix any function $\tilde{u}(x,t) \in H_{1,1}(Q, S \bigcup \Omega)$ and consider the initial-boundary value problem for linear pseudoparabolic equation

$$-(m^{ij}(\tilde{u})u_{tx_j})_{x_i} + m(\tilde{u})u_t - (l^{ij}(x,t)u_{x_j})_{x_i} + l(x,t)u = f(x,t) + f^i_{x_i}(x,t), \quad (2.4)$$

$$u|_S = 0, \quad (2.5)$$

$$u|_{t=0} = 0. \quad (2.6)$$

As it was shown in [3] the weak solution $u(x,t)$ of problem (2.4) – (2.6) exists in Q and this solution satisfies the inequalities

$$||u||_{H_{1,1}(Q)} \leq C, \quad (2.7)$$

where C is a constant dependent only on $Q, m_{00}, f, f^i (i = 1, \ldots, n)$ and independent of $\tilde{u}(x,t)$.

Let D be the set of functions from $H_{1,1}(Q, S \cup \Omega)$ which satisfy the estimation (2.7)

$$D = \{v \in H_{1,1}(Q, S \cup \Omega) : \ ||v||_{H_{1,1}(Q)} \leq C\}$$

Let us define operator $T : D \to D$ in the following way: if the function $\tilde{u}$ belongs to D, then the solution u of considered problem (2.4) – (2.6) is $u = T\tilde{u}$. The operator T is weakly continuous, i.e. if $\tilde{u}^n \rightharpoonup \tilde{u}$ then $T\tilde{u}^n \rightharpoonup T\tilde{u}$, and therefore the conditions of the second Shauder principle (fixed point theorem) are fulfilled . It means, that the weak solution of problem (1.1) with homogeneous initial condition exists.

Let us consider now the problem (1.1) with a nonhomogeneous initial condition. This problem can be reduced to problem with homogeneous initial condition for function $g(x,t) = u(x,t) - \varphi(x) \in H_{1,1}(Q, S \bigcup \Omega)$. ■

Theorem 2. (Uniqueness). *Suppose assumptions* (A1) – (A9) *are satisfied and conditions*

$$|m(x) - m(y)| \leq \alpha|x - y|, \quad |m^{ij}(x) - m^{ij}(y)| \leq \alpha|x - y|$$

are satisfied for all $x, y \in R$*. If* $u(t,x)$ *is a weak solution of the problem* (1.1), *and* $u_t, u_{tx_i} \in Lp$ $(i = 1, ..., n)$*, where* $p > 2$*, if* $n = 2$*, and* $p \geq n$*, if* $n > 2$*, then the problem* (1.1) *can have only one solution.*

Proof. Let us suppose that $u(x,t)$ and $\tilde{u}(x,t)$ are solutions of problem (1.1). Let us denote

$$u - \tilde{u} = w \in H_{1,1}(Q),$$

$$Q_\tau = Q \cap \{t < \tau\}, \quad \tau \in [0;T], \quad F(\tau) = \Big(\int\limits_{Q_\tau} \sum_{i=1}^{n} w^2_{tx_i}\, dxdt\Big)^q,$$

where $\frac{1}{2} + \frac{1}{2q} + \frac{1}{p} = 1$. Using the definition of the weak solution, for functions $u(x,t)$ and $\tilde{u}(x,t)$ from the integral identity (2.3), where $v = w_t e^{-2\mu t}$, it is possible to show, that

$$F(\tau) \le C \int_0^\tau F(t)\, dt.$$

From this follows that $w = 0$ in Q and $u = \tilde{u}$. ■

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