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FINITE ELEMENT MODEL UPDATING AND PARTIAL EIGENVALUE ASSIGNMENT IN STRUCTURAL DYNAMICS: RECENT DEVELOPMENTS ON COMPUTATIONAL METHODS

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Abstract. This paper gives a brief overview of some of the recent developments on computational methods for several inverse eigenvalue problems for matrix quadratic pencils arising in finite-element model updating and feedback control designs for vibrating structures.
Key words: Finite Element, Partial Eigenvalue Assignment

1. Introduction

The *Millennium* bridge on the river Thames in London was closed after two days of its inauguration in June 2000, because, on its opening day it started to wobble due to the weights of thousands of people who rushed to see the bridge on the first day. This is an example of a dangerous vibration phenomenon, called *resonance*, which occurs when some natural frequencies of the vibrating systems become close to those of external forces. In this case, the external force was the weights of the human bodies. Other classical examples of resonance include fall of the *Tacoma* bridge in the state of Washington in USA, and the of the *Broughton* bridge in England. In the case of Tacoma bridge, the external force was gusty wind, and in the case of the Broughton bridge, it was the weight of the soldiers marching in the bridge. Thus, mathematically, a resonance problem is the one of reassigning the few resonant frequencies (eigenvalues) to desired locations, while keeping the remaining large number of them and corresponding eigenvectors unchanged.

Such problems, when solved using feedback control, are called *Quadratic Partial Eigenvalue Assignment* and *Partial Eigenstructure Assignment* problems (QPEVAP and QPESAP). The use of feedback, unfortunately, destroys the symmetry. While control applications do not demand that the updated model remains symmetric [15, 25], there are other practical problems, such as the *Model Updating Problem* (MUP), for which preservation of symmetry and other properties is crucial. The Model Updating Problem is a practical industrial problem and arises in aerospace, manufacturing, automobile, and other vibration industries. In these industries, a theoretical finite-element generated symmetric model often needs to be updated using a few measured eigenvalues and eigenvectors, obtained form a practical structure in such a way that the updated model can be used with confidence for future designs [10, 13].

For useful viable practical applications, these problems should be solved satisfying certain mathematical, computational, and engineering requirements. These requirements include:

- The updated model should be physically meaningful and be related to physical changes to finite elements in the original model.
- The problems should be solved without a *priori model reduction* and/or transformation to a convenient form, such as to a standard first-order problem; because, some important properties of the model might be completely destroyed during these procedures, and furthermore, these procedures may be computationally dangerous.
- The invariance of the part of the spectrum and the eigenvectors, as required by the problems, should be ascertained with mathematical results, since it is impossible to verify these in a computational or experimental setting, because of the very high order nature of these models.
- The computational algorithms should use only a small subset of the eigenvalues and eigenvectors. This is because the underlying quadratic eigenvalue problem is nonlinear and the state-of-the-art techniques are capable of computing only a few extremal eigenvalues and eigenvectors [12, 26]. The computational schemes should also be able to take advantage of the exploitable properties such as the sparsity, bandness, positive definiteness, etc., often offered by the mathematical models.

In current engineering practice, most methods, especially those for the MUP fail to satisfy these requirements. In particular, the hardest part of each of these problems, namely, keeping a large part of the spectrum and eigenvectors invariant in the updated model, is hardly considered. It is only hoped that they will not be severely affected by an update.

In the last few years, the author and his collaborators have developed a novel approach, meeting most of the above mentioned challenges, for QPEVAP and QPESAP. In particular, it works exclusively with the quadratic pencil without any a priori model reduction or transformation to standard first-order problem and mathematically guarantees the invariance of the required spectrum and the eigenvalues.

A similar approach has also been developed for the model updating problem. It is to be stressed in this context that there is a fundamental difference between the QPEVAP and QPESAP. The MUP requires that the updated model remains symmetric and use of feedback control is not a requirement.

Two new numerically viable methods and the supporting mathematical theories have been developed in the Ph.D dissertation of J. Carvalho [3] and in [5]. Both the methods are capable of meeting the two toughest requirements of the problem: preserving the symmetry and retaining the large number of eigenvalues and eigenvectors that do not participate in updating. Furthermore, unlike all the previous methods, neither requires model reduction or expansion of the eigenvectors.

In spite of these new developments, the problem has not been fully solved. The method in [3] solves the problem in case of an undamped model, and the other [5] solves a related problem on symmetric eigenvalue embedding. Although not originally designed to solve the MUP, it does constitute a solution when only the measured spectrum (but not eigenvectors) is considered for updating.

The purpose of this paper is to present a brief overview of the above recently developed techniques for the inverse problems under considerations. The emphasis is on the clear presentations of the algorithms for better and easy understanding of the practicing engineers and highlighting the engineering and computational advantages and disadvantages.

2. Problem Formulation, Background and Motivations

2.1. Model updating problem

A finite-element model of a vibrating structure can be represented as

$$M\ddot{(t)} + C\dot{x}(t) + Kx(t) = 0$$

where M, C, and K, each of order n, are called, respectively, the *mass, stiffness* and *damping* matrices. In general, it is assumed that

$$M = M^T > 0, \;\; K = K^T \geq 0, \;\; C = C^T.$$

Assuming a solution $x(t) = ue^{\lambda t}$, one obtains the *quadratic eigenvalues problem*:

$$(\lambda^2 M + \lambda C + K)u(t) = 0.$$

The matrix $P(\lambda) = \lambda^2 M + \lambda C + K$, called the quadratic matrix pencil, and it has $2n$ eigenvalues and $2n$ eigenvectors.

Let $\{\lambda_1, ..., \lambda_p; \lambda_{p+1}, \lambda_{2n}\}$ and $\{x_1, ..., x_p; x_{p+1}, x_{2n}\}$ be the eigenvalues and eigenvectors of $P(\lambda)$. Suppose that only a small subset p ($p << 2n$), say $\{\lambda_1, ..., \lambda_p\}$ and $\{x_1, ..., x_p\}$, are computed. Let $\{\mu_1, ..., \mu_p\}$ and $\{v_1, ..., v_p\}$ be the corresponding measured ones from a real-life structure. Very often in practice, the eigenvalues and eigenvectors of this theoretical finite-element generated matrix pencil do not match well with those of measured ones.

A vibration engineer then needs to update this model so that the inaccurate modeling assumptions can be corrected in the original model and the updated model can the be used in future designs with confidence. In the context of aerospace applications (such as those at the Boeing company), the measured data are generated from a ground vibration test, and the updated model is used in the future for flutter analysis.

Mathematically, the problem might be defined as follows:

MUP: Given the symmetric finite-element quadratic pencil (M, C, K), with a set of computed eigenpairs $\{\lambda_k, x_k\}, k = 1, ..., p$ and a set $\{\mu_k, v_k\}, k = 1, ..., p$ of measured eigenvalues and eigenvectors, find an updated symmetric pencil (M_U, C_U, K_U) such that:

(i) The subset $\{\lambda_k, x_k\}$ is replaced by $\{\mu_k, v_k\}, k = 1, ..., p$ as eigenvalues and the corresponding eigenvectors of the updated pencil (M_U, C_U, K_U),
(ii) The remaining subset of $2n - p$ eigenvalues and the corresponding eigenvectors of (M_U, C_U, K_U) remain the same as those of (M, C, K).

Remark 1. The requirement (ii) is known as the *no spill-over* phenomenon in the engineering literature. Satisfaction of this requirement guarantees that the modal parameters, not related to the physical measurements, will not change.

2.2. Quadratic partial eigenvalue and eigenstructure assignment problems

Suppose that a control force of the form $Bu(t)$ is applied to the vibrating structure, where B is the control matrix and $u(t)$ is the control vector. Choosing $u(t) = F_1\dot{x}(t) + F_2x(t)$, we obtain the second-order closed-loop system

$$M\ddot{(t)} + (C - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0$$

with the associated quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2).$$

as the *closed-loop* pencil. Let $\{\lambda_1, ..., \lambda_p\}$ be a self-conjugate set of resonant eigenvalues and $\{x_1, ..., x_p\}$ be the corresponding eigenvectors of $P(\lambda)$, as defined above.

Suppose that the stiffness and damping matrices are to be modified using the control matrix $B \in \boldsymbol{R}^{n\times m}$ $(m \leq n)$ such that the set $\{\lambda_1, ..., \lambda_p\}$ is replaced by another self-conjugate set, $\{\mu_1, ..., \mu_p\}$, leaving the remaining $(2n - p)$ eigenvalue unchanged. This gives rise to:

QPEVAP: Find real feedback matrices F_1 and F_2, of each order m$\times n$, such that the closed-loop pencil $P_U(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ has the spectrum $\{\mu_1, ..., \mu_p; \lambda_{p+1}, ..., \lambda_{2n}\}$.

While the eigenvalues determine the rate at which the system response decays or grows, the eigenvectors determine the shape of response. Thus, the reassignment of both a set of eigenvalues and the eigenvectors should be considered. Unfortunately, the problem in this case may not be solvable if the control matrix B is given a prior [1]. This leads to:

QPESAP: Find a real control matrix B of order $n \times m$ $(m < n)$ and real feedback matrices F_1 and F_2 such that the spectrum of the updated pencil $P_U(\lambda) = \lambda^2 M + \lambda(C - BF_1) + (K - BF_2)$ is the set $\{\mu_1, ..., \mu_p; \lambda_{p+1}, ..., \lambda_{2n}\}$ with $\{x_{c_1}, x_{c_2}, ..., x_{c_p}; x_{p+1}, ..., x_{2n}\}$ as the associated eigenvectors.

3. Existing Methods and their Drawbacks

The model updating problem (**MUP**) is of immense practical importance and arises in aerospace, automobile, and other vibration industries. The problem, therefore, has been very widely studied and many results and methods exist (see [8, 10, 13]). The problem remains unsolved and the existing methods have severe computational and engineering limitations, which restrict their usefulness in practical applications.

Most of the existing methods deal with updating the linear pencil $K - \lambda M$, rather than the quadratic pencil $P(\lambda)$. Unfortunately, even in this simpler case, the methods fail to produce updated pencils that are physically meaningful. These methods preserve the symmetry and the measured eigenvalues, and eigenvectors are incorporated rather accurately into the updated model; however, they *"can not guarantee that extra, spurious modes are not introduced into the range of the frequency range of interest"*([13], pp.127). In general, none of these methods is capable of completely retaining the eigenvalues and eigenvectors of the model that are not to be affected by updating.

There are a few other methods [16, 17, 28] that consider damping. Thus, they actually work with the quadratic pencil itself. These methods come in two stages.

In stage I, the measured eigenvalues and eigenvectors are incorporated into the model using feedback control. Unfortunately, in the process of doing so, the symmetry is destroyed.

In stage II, an optimization technique is used to recover the symmetry.

The difficulties with these methods are:

(i) The feedback control techniques are not usually numerically viable,

(ii) Though the optimization techniques improve the symmetry, this *" will not always make the resulting damping and stiffness matrices symmetric"* ([13], p.152);

(iii) No spill-over can be guaranteed.

Another practical difficulty with all these methods is that either a *model reduction technique* has to be applied to the FEM or *measured mode shapes have to be expanded* to the full length of the FEM. This is because the data (eigenvectors) measured experimentally from a real-life structure are very often incomplete in the sense that, due to hardware limitations, it can be measured only at a subset of the degrees of freedom of the finite element model. While the finite element models can be of several thousand degrees of freedom, the measured eigenvectors from a practical structure can be of lengths of, at most, a couple hundred. Therefore, comparing the analytical eigenvectors with those from a real-life structure that has different lengths becomes impossible in practice unless some measures are taken to fill-in the missing entries or the order of the model is reduced. Although good algorithms for model reduction in first-order state-space systems exist [2, 9, 14], model-reduction algorithms that work directly in matrix second-order models are virtually non-existent.

An obvious way to solve the quadratic eigenvalue assignment and related problems is to transform such a problem to a standard first-order problem, for which there exists excellent numerical methods [6, 22], including some for partial [7, 23] and robust and minimum norm assignments [18, 19, 20, 21, 27]. This technique might need an ill-conditioned matrix inversion or it will lead to a descriptor system problem, the methods for which are not well-developed. Furthermore, some of the important properties, offered by practical problems, such as positive definiteness, bandness, and sparsity, will be destroyed.

The Independent Modal Space Control (IMSC) approach, popular in engineering literature, aims at solving these problems in quadratic setting by decoupling, but unfortunately, it requires the knowledge of the complete spectrum of the quadratic pencil and the eigenvectors for implementation. In addition, a stringent requirement on the commutativity of the coefficient matrices has to be satisfied [15]. Above all, to handle large and sparse problems, which naturally arise in structural dynamics and other vibration applications, the order of the model has to be reduced by using a model reduction technique.

3.1. New results on MUP

The author and his collaborators have developed two new methods for the MUP.

Model updating using incomplete data

This method stated below has the practical significance that *it can deal with the problem of incomplete measured data, described in the previous section, without a priori model reduction or expansion of the eigenvectors.* First, the mathematical results that *grantee the existence of solution of the MUP* in the case when damping is assumed to be zero are stated below. Consider the undamped symmetric positive semidefinite quadratic pencil $P(\lambda) = \lambda^2 M + K$. Let $\Lambda_1 =$ diag $(\lambda_1, \lambda_2, ..., \lambda_p)$ and X_1 be the corresponding eigenvector matrix. Let Σ_1^2 be a matrix containing information about the measured eigenvalues and Y_{11} be the corresponding eigenvector matrix consisting only of the first m measured components. Let $\Lambda_2 = \text{diag}(\lambda_{p+1}, ..., \lambda_{2n})$.

Theorem 1. ***(Spectrum and eigenvector invariance by updating).*** *Assume that Λ_1 and Λ_2 do not have a common eigenvalue. Then, for every symmetric matrix ϕ, the updated pencil $P_U(\lambda) = \lambda^2 M + K_U$, where*

$$K_U = K - MX_1\phi X_1^T M$$

is such that $(2n-p)$ eigenvalues $\lambda_{p+1}, ..., \lambda_{2n}$ and the corresponding eigenvectors of this pencil are the same as those of $P(\lambda)$.

The challenge now is to choose the matrix ϕ in such a way that

(i) the missing components of the measured eigenvector matrix will be appropriately computed,

(ii) the updated pencil will also contain the measured eigenvalues and eigenvectors.

To this end, the following result is proved:

Partition $M = [M_1, M_2]$ and $K = [K_1, K_2]$, where $M_1, K_1 \in \boldsymbol{R}^{n\times p}$. Let $Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}$, where Y_{12} is the missing part of the measured eigenvectors. Let

$$MX_1 = (U_1, U_2)\begin{pmatrix} Z \\ 0 \end{pmatrix}$$

be the QR factorization of MX_1. It is shown that, if Y_{12} is found by solving the Sylvester equation:

$$U_2^T M_2 Y_{12}(\Sigma_1^2)^2 + U_2^T K_2 Y_{12} = -U_2^T(K_1 Y_{11} + M_1 Y_{11}(\Sigma_1^2)^2),$$

followed by computing ϕ satisfying

$$(Y_1^T M X_1)\phi(Y_1^T M X_1)^T = Y_1^T M Y_1(\Sigma_1^2)^2 + Y_1^T K Y_1,$$

and this ϕ is used in forming K_U in Theorem *3.1*, then the spectrum of the updated pencil is:

$$\Omega(\lambda^2 M + K_U) = \{\mu_1, ..., \mu_p;\ \lambda_{p+1}, ..., \lambda_{2n}\}.$$

Theorem 1 along with the above result constitute a complete solution in the undamped case ($C = 0$) and is the *state-of-the-art* result on this problem.

Algorithm 1. Modeling Updating of an Undamped Symmetric Positive Semidefinite Model using Incomplete Measured Data

Step 1: Form the matrices $\Sigma_1^2 \in \boldsymbol{R}^{m\times m}$ and $Y_{11} \in \boldsymbol{R}^{m\times m}$ from the available data. Form the corresponding matrices $\Lambda_1^2 \in \boldsymbol{R}^{n\times m}$ and $X_1 \in \boldsymbol{R}^{n\times m}$.

Step 2: Compute the matrices $U_1 \in \boldsymbol{R}^{n\times m}, U_2 \in \boldsymbol{R}^{n\times(n-m)}$ and $Z \in \boldsymbol{R}^{m\times m}$ from the QR factorization:

$$MX_1 = [U_1\ U_2]\begin{bmatrix} Z \\ 0 \end{bmatrix}.$$

Step 3: Partition $M = [M_1\ M_2],\ K = [K_1\ K_2]$ where $M_1,\ K_1 \in \boldsymbol{R}^{n\times m}$.

Step 4: Solve the following equation to obtain $Y_{12} \in \boldsymbol{R}^{(n-m)\times m}$:

$$U_2^T M_2 Y_{12}\Sigma_1^2 + U_2^T K_2 Y_{12} = -U_2^T[K_1 Y_{11} + m_1 Y_{11}\Sigma_1^2]$$

and form the matrix

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}.$$

Step 5: Compute the matrix $L \in \boldsymbol{R}^{m\times m}$ and the diagonal matrix $J \in \boldsymbol{R}^{m\times m}$ such that $LJL^T = Y_1^T M Y_1$ is a symmetric (LDL^T) factorization of $Y_1^T M Y_1$. Update the matrix Y_1 by $Y_1 \leftrightarrow Y_1(L^{-1})^T$.

Step 6: Compute $\Psi \in \boldsymbol{R}^{m\times m}$ by solving the following system of equations:

$$(Y_1^T M X_1)\Psi(Y_1^T M X_1)^T = Y_1^T M Y_1(\Sigma_1)^2 + Y_1^T K Y_1$$

Step 7: Update

$$\tilde{K} = K - M X_1 \Psi X_1^T M.$$

Remark 2. The algorithm above can also be used when a complete measured data is available. In this case, Steps 2, 3 and 4 must be skipped. We recommend doing this when the measurements of Σ_1 and Y_1 are not highly accurate.

Example 1. Model Updating using Incomplete Measured Data

Consider the positive semidefinite model (M, D, K) where $D = 0$ and matrices M and K are given by

$$M = \begin{bmatrix} 1.2940 & 0. & 0. & 0. & 0. \\ 0. & 1.2940 & 0. & 0. & 0. \\ 0. & 0. & 1.2940 & 0. & 0. \\ 0. & 0. & 0. & 1.2940 & 0. \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

$$K = \begin{bmatrix} 1188.5000 & 196.6000 & 0. & 0. & -642.4000 \\ 196.6000 & 626.3000 & 0. & -555.6000 & 0. \\ 0. & 0. & 1188.5000 & -196.6000 & -546.1000 \\ 0. & -555.6000 & -196.6000 & 626.3000 & 196.6000 \\ -642.4000 & 0. & -546.1000 & 196.6000 & 4019.1000 \end{bmatrix}$$

This model has two infinite eigenvalues.

Step 1. The matrices of measured frequencies and mode shapes are taken as:

$$\Sigma_1^2 = \begin{bmatrix} -23.5500 & \\ & -990.0800 \end{bmatrix}, \quad Y_{11} = \begin{bmatrix} 0.3000 & 1.2000 \\ 0.3500 & -1.1300 \end{bmatrix}.$$

The corresponding modal matrices are:

$$\Lambda_1^2 = \begin{bmatrix} -23.6929 & \\ & -991.1000 \end{bmatrix}, \quad X_1 \begin{bmatrix} 0.2045 & -0.5069 \\ -1. & -0.9986 \\ -0.1403 & -0.7470 \\ -0.9997 & 1 \\ 0.0625 & -0.2314 \end{bmatrix}.$$

Also, note that

$$\Lambda_2^2 = \begin{bmatrix} -702.3357 & \\ & -943.6921 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1. & 1. \\ 0.5775 & 0.2627 \\ -0.9602 & -0.9325 \\ -0.6475 & 0.0726 \\ -0.2586 & 0.0296 \end{bmatrix}.$$

Step 2. From the QR factorization of MX_1:

$$U_1 = \begin{bmatrix} -0.1424 & -0.3023 \\ 0.6966 & -0.5955 \\ 0.0977 & -0.4455 \\ 0.6964 & 0.5963 \\ 0. & 0. \end{bmatrix}, \quad U_2 = \begin{bmatrix} -0.1668 & 0.9276 & 0. \\ -0.3696 & -0.1536 & 0. \\ 0.8895 & 0.0298 & 0. \\ 0.2108 & 0.3392 & 0. \\ 0. & 0. & 1. \end{bmatrix}$$

and

$$Z = \begin{bmatrix} -1.8576 & 0 \\ 0 & 2.1700 \end{bmatrix}.$$

Step 3. The partition of matrices M and K is straightforward.
Step 4. The solution of the descriptor Sylvester equation is

$$Y_{12} = \begin{bmatrix} 0.3662 & 0.6685 \\ -0.6942 & -3.8523 \\ 0.1317 & 0.4711 \end{bmatrix}.$$

Step 5. Computing the LDL^T factorization and updating Y_1:

$$L = \begin{bmatrix} 1 & 0 \\ 3.4804 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1.0722 & 0 \\ 0 & 10.3107 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0.3000 & 0.1559 \\ 0.3500 & -2.3481 \\ 0.3662 & -0.6061 \\ -0.6942 & -1.4362 \\ 0.1317 & 0.0128 \end{bmatrix}.$$

Step 6. The symmetric matrix Ψ is

$$\Psi = \begin{bmatrix} -335.7777 & -80.9068 \\ -80.9068 & 246.3905 \end{bmatrix}.$$

Step 7. The updated stiffness matrix is:

$$\tilde{K} = \begin{bmatrix} 1077.9001 & -86.2129 & -183.4235 & 190.5648 & -642.4000 \\ -86.2129 & 1047.6955 & -108.7290 & 418.2325 & 0. \\ -183.4235 & -108.7290 & 997.7215 & 504.7781 & 196.6000 \\ -642.4000 & 0. & -546.1000 & 196.6000 & 4019.1000 \end{bmatrix}.$$

Verification:

$$||MY_1\Sigma_1^2 + \tilde{K}Y_1||_F = 3.1607 \times 10^{-13},$$

$$||MX_2\Lambda_2^2 + \tilde{K}X_2||_F = 2.4177 \times 10^{-12}.$$

Therefore, we conclude that

- The incomplete measured data was entirely and accurately incorporated in the new model.
- The unmeasured frequencies and mode shapes *did not* change, and therefore the update did not produce spill-over.

Model updating by eigenvalue embedding

The eigenvalue embedding is a process of embedding a set of measured eigenvalues in a symmetric FEM in such a way the updated model is still symmetric and the other eigenvalues and eigenvectors of the original model do not change. Thus, an eigenvalue embedding method solves the model updating problem in the case when the measured eigenvectors are not considered for updating.

Two eigenvalue embedding methods have been proposed so far in the literature: a method by Ferng, Lin, Pierce and Wang [11] using theory of nonequivalence transformation of matrix polynomials, and a method by the author and collaborators [4]. The first paper considers updating of a slightly more general model that arises in the construction of aircraft structure model for dynamic load analysis, but unfortunately, the symmetry of the model is not preserved. Ours is a *symmetry preserving method.* We state below our method in a simple form that assigns just one isolated real eigenvalue. However, embedding a pair of complex eigenvalues and simultaneously embedding a group of real eigenvalues can also be done, and the relevant algorithms and theory appear in [4].

Algorithm 2. **Algorithm for model updating by eigenvalue embedding**

Given the symmetric positive definite FEM (M, D, K), a finite element generated real eigenpairs (λ_1, y_1), and a measured eigenvalue μ_1 such that

$$y_1^T K y_1 = 1, \quad 1 - \lambda_1 \mu_1 (y_1^T M y_1) \neq 0, \quad 1 - \lambda_1^2 (y_1^T M y_1) \neq 0.$$

Step 1. Compute

$$\theta_1 = y_1^T M y_1, \epsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \theta_1}.$$

Step 2. Form the symmetric rank-one updates:

$$M_U = M - \epsilon_1 \lambda_1 M y_1 y_1^T M, \quad K_U = K - \frac{\epsilon_1}{\lambda_1} K y_1 y_1^T K,$$

$$C_U = C + \epsilon_1 (M y_1 y_1^T K + K y_1 y_1^T M).$$

Theorem 2. *(Symmetric eigenvalue embedding). The updated pencil* $P_U(\lambda) = \lambda^2 M_U + K_U + C_U$ *is symmetric and has the following properties:*

(i) The number μ_1 *is an eigenvalue of* $P_U(\lambda)$*,*

(ii) $(\lambda_k, x_k),\ k = 2, ..., 2n$ *are also the eigenpairs of* $P_U(\lambda)$*.*

3.2. New Results on QPEVAP, and QPESAP, and QRPEVAP

In this section, we present our *direct* and *partial-modal* which of the PQEAP. It is direct in the sense that the solution is obtained directly in the quadratic setting without any a priori transformation to a first-order problem. It is partial-modal, since it requires knowledge only those small number of resonant eigenvalues and the corresponding eigenvector. The other distinguished features of this algorithm include:

(i) computational requirements are minimal,

(ii) the structures of the mass, stiffness, and damping matrices can be exploited in the computational setting and furthermore, no a *priori model reduction* is necessary.

Above all, the *no spill-over* property is guaranteed with a mathematical result. Similar result exists for QPEASP [24]. The above features make the algorithms practical even for very large and sparse problems.

4. Notations

$\Omega(P(\lambda))$ is spectrum of $P(\lambda) = \{\lambda_1, ..., \lambda_p; \lambda_{p+1}, ..., \lambda_{2n}\}$, X the right eigenvector matrix of $P(\lambda)$, Y the left eigenvector matrix of $P(\lambda)$, A^H Hermitian conjugate of A, $\Lambda_1 =$ diag $(\lambda_1, ..., \lambda_p)$, $\Lambda_2 =$ diag $(\lambda_{pH}, ..., \lambda_{2n})$, $Y_1 = (y_1, y_2, ..., y_p)$, the left eigenvector matrix of $P(\lambda)$ associated with $\lambda_1, ..., \lambda_p$.

Algorithm: An Algorithm for QPEAP
Assumptions:
(i) $\{\lambda_1, ..., \lambda_p\}$ is self-conjugate.
(ii) $\{\lambda_1, ..., \lambda_p\} \cap \{\lambda_{p+1}, ..., \lambda_{2n}\} = 0$.
(iii) $(P(\lambda), B)$ is partially controllable with respect to $\{\lambda_1, ..., \lambda_p\}$.

Theorem 3. *Let the above assumptions hold and let Φ be any arbitrary matrix. (a) Then the matrices F_1 and F_2 defined by $F_1 = \Phi Y_1^H M$ and $F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C)$ are such that $2n - p$ eigenvalues $\{\lambda_{p+1}, \dots, \lambda_{2n}\}$ of the closed-loop pencil $P_c(\lambda)$ remain unchanged, (b) If Φ in part (a) is chosen solving the $p \times p$ linear system $QZ_1 = \Gamma$, where Γ is arbitrary and Z_1 is the unique solution of the sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H B \Gamma,$$

then F_1 and F_2 defined in Part (a) will solve the QPEAP.

Algorithm 3. **An Algorithm for QPEAP**:

Given M, C, K, Λ, Y_1 and Λ_{cl} (as defined above) the following algorithm, under the above assumptions, compute F_1 and F_2 that solve the QPEAP.

Step 1. Choose Γ arbitrary and solve the $p \times p$ Sylvester equation for Z_1 :

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H B \Gamma.$$

Step 2. Solve the $p \times p$ linear algebraic system for

$$Z_1 : \Phi Z_1 = \Gamma.$$

Step 3. Compute F_1 and F_2 as defined above.

We note that

(a) The existence of the unique solution Z_1 in step 1 is guaranteed by assuming that $\{\lambda_1, \ldots, \lambda_p\}$ and $\{\mu_1, \ldots, \mu_p\}$ are disjoint. The assumption is practical because, it does not make any sense to reassign an eigenvalue which is undesirable.

(b) The assumption (ii) implies the existence of Z_1.

5. Conclusion

A brief account of some of the recently developed new algorithms for three important practical inverse eigenvalue problems, namely Quadratic Partial Eigenvalue Assignment and Eigenstructure Assignment, and Finite Element Model Updating problems arising in vibration and control engineering are presented here. The direct and partial modal nature and minimal computational requirements of these algorithms make them attractive for practical applications even when the structures are very large. It is hoped that these results will provide much incentive for researchers in vibration engineering to solve many other vibration control problems in a practical way. It is noted in this context that research in vibration engineering still lags other areas of control and much remains to be done.

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