Mathematical Modelling and Analysis 2005. Pages 361–366
Proceedings of the 10th International Conference MMA2005&CMAM2, Trakai

MODELLING OF GEOMETRIC ANISOTROPIC SPATIAL VARIATION

L. BUDRIKAITĖ and K. DUČINSKAS

Klaipėda University
Faculty of Natural and Mathematical sciences

H.Manto 84, Klaipėda, Lithuania

E-mail: linabudrikaite@lycos.com; duce@gmf.ku.lt

Abstract. Isotropic processes form an inadequate basis in modelling many spatially distributed data. In particular environmental phenomena often have strong anisotropic spatial variation, especially when the regions monitored are very large (see [1]). Among different forms of spatial anisotropy a geometric anisotropy is most common (see [4]). Geometric anisotropy, which provides the most common generation of isotropy within stationarity, is typically dealt with by simple transformations of coordinates. For modelling spatial processes, we propose a rich class of stationary geometric anisotropic variograms.

Objective of our investigation is to select and identify optimal models of isotropic variograms for different direction regions, using R, a system for statistical computation and graphics [2]. Spatial data was used for realization of the proposed modelling procedure. General form of geometric anisotropic semivariogram for salinity data was obtained.

Key words: semivariogram, geometric anisotropy, sill, range, nugget

1. Introduction

Spatially independent data show the same variability regardless of the location of data points. However, spatial data in most cases are not spatially independent. Data values, which are close spatially, show stronger correlation than data values, which are farther away from each other.

The traditional measurement of spatial correlation is the semivariogram, commonly called the variogram. For spatial locations $\left\{s_i : s_i \in D \subset R^d\right\}$ in a region D, suppose we observe responses $Z(s_i), i = 1,...,N$, where $Z = (Z(s_1), Z(s_2), ..., Z(s_N))\acute{}$ is viewed as a single observation from a random field. Under the intrinsic hypothesis of Matérn (1963) we have:

$$E(Z(s_1) - Z(s_2)) = 0, \quad Var(Z(s_1) - Z(s_2)) = 2\gamma(s_1 - s_2) = 2\gamma(h), \quad (1.1)$$

where $h = s_1 - s_2$ is separation vector, $2\gamma(h)$ is called the variogram and $\gamma(h)$ is the semivariogram. A stronger assumption is that the process $Z(s)$ is a second-order or weakly stationary, i.e.:

$$E(Z(s)) = \mu \text{ and } Cov(Z(s_1), Z(s_2)) = C(s_1 - s_2) = C(h) < \infty. \qquad (1.2)$$

Classical semivariogram has the following shape as shown in Figure 1. Range (Θ_2) is the distance at which the semivariogram becomes a constant.

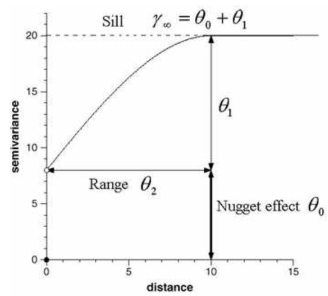


Figure 1. Semivariogram representation.

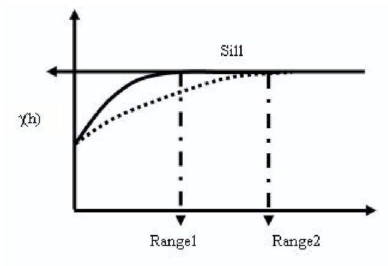


Figure 2. Geometric anisotropy.

If $\lim_{|h|\to\infty} \gamma(|h|) = \gamma_\infty < \infty$, then γ_∞ is called sill of semivariogram. The nugget effect shows the pure random variation in population density or it may be associated with sampling error. If $\gamma(|h|) \to \Theta_0 > 0$ when $|h| \to 0$, then Θ_0 is called the nugget effect [3].

A semivariogram is anisotropic if it changes in some way with respect to direction. If value of semivariogram depends only on length of vector h, then we have isotropic semivariogram. If value of semivariogram depends not only on length of vector h, but depends also on direction of vector then we have anisotropic semivariogram.

Semivariogram modeling is the foundation for geostatistical analysis – in order to apply kriging to a data set it is necessary to model the variogram. Theoretical variogram models for kriging are based on isotropic models, so correction for any anisotropies is necessary to use kriging methodology.

There are two types of anisotropy: geometric and zonal anisotropy. Geometric anisotropy occurs when the range, but not the sill, of the semivariogram changes in different directions (see Figure 2). Zonal anisotropy exists when sill of semivariogram change with direction. We shall focus on geometric anisotropy.

Geometric anisotropy means that the correlation is stronger in one direction than it is in the other directions. Mathematically, if one plots the directional ranges, in two dimensions they would fall on the edge of an ellipse (see Figure 3a) and in three-dimensional case they would fall on the surface of an ellipsoid (see Figure 3b), where major and minor axes of ellipse/ellipsoid correspond to the largest and shortest ranges of directional semivariograms.

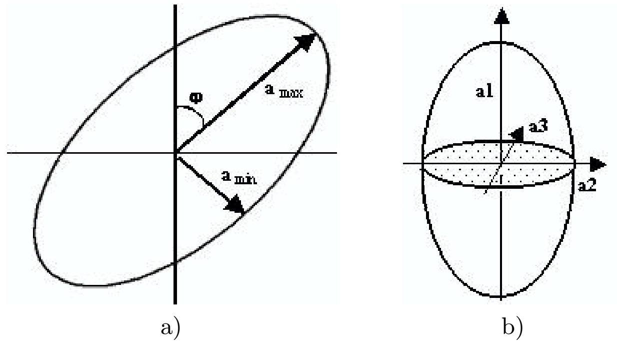


Figure 3. Directional ranges: a) two-dimensional case, b) three-dimensional case.

One way in which geometric anisotropy can be identified is by calculating and plotting experimental directional semivariograms. Differences in sample variograms computed using different angles could be an indication of anisotropy.

Geometric anisotropy can be modeled by changing the variogram model for an isotropic process transforming the coordinates

$$\gamma(s_i - s_j) = \gamma(\|A(s_i - s_j)\|),$$

where A is transformation matrix. The basic procedure consists of 4 steps. We are going to describe two-dimensional and three-dimensional cases.

Two-dimensional case

First step is to identify axes of anisotropy. They can be detected generating a focused experimental variogram in several different directions and observing whether or not there are significant differences in the resulting variograms. Usually we study angles (φ) of 0^o, 45^o, 90^o, and 135^o with tolerance angle $\epsilon = 45^o$. Then each angle group (α_i) is defined as $\varphi - \frac{\epsilon}{2} < \alpha_i < \varphi + \frac{\epsilon}{2}$ [3]. If anisotropy exists, the ranges or sills of the two variograms will differ.

Once anisotropy has been detected the second step is to rotate data axes to match axes of anisotropy. This can be done using rotation matrix

$$R = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix},$$

where φ is an angle from y (North direction) to the major axis of ellipse.

Third step is reduction directional variograms to a single variogram with standardizes range of 1. Distance transformation can be represented as matrix

$$T = \begin{pmatrix} 1/a_{max} & 0 \\ 0 & 1/a_{min} \end{pmatrix}. \qquad (1.3)$$

where a_{max} is major range of anisotropy ellipse, a_{min} is minor range of anisotropy ellipse.

Finally we need to combine these rotation and distance scaling transformation matrixes and we get transformation matrix $A = TR$.

So overall model is

$$\gamma(s_i - s_j) = \gamma(\|TR(s_i - s_j)\|). \tag{1.4}$$

Three dimensional case

A semivariogram structure with geometric anisotropy in a 3-D space corresponds to a tri-axial ellipsoid (see Figure 3b). Procedure of identification of anisotropy axes is the same as in two dimensional case.

Six parameters are needed to specify the ellipsoid: three ranges $(a1, a2, a3)$ and three rotation angles to quantify orientation in a coordinate system.With these six parameters we can transform an anisotropic variogram structure into an isotropic one. Anisotropic variogram structure appears as an ellipsoid, after the change of coordinates it becomes a unit sphere.

Each rotation rotates a plane formed by two coordinates about the third coordinate by some angle. Each rotation has its own rotation matrix, so final rotation matrix R can be combined as follows

$$R = \begin{bmatrix} \cos\varphi_2\cos\varphi_1 & \cos\varphi_2\sin\varphi_1 & -\sin\varphi_2 \\ -\cos\varphi_3\sin\varphi_1 + \sin\varphi_3\sin\varphi_2\cos\varphi_1 & \cos\varphi_3\cos\varphi_1 + \sin\varphi_3\sin\varphi_2\sin\varphi_1 & \sin\varphi_3\cos\varphi_2 \\ \sin\varphi_3\sin\varphi_1 + \cos\varphi_3\sin\varphi_2\cos\varphi_1 & -\sin\varphi_3\cos\varphi_1 + \cos\varphi_3\sin\varphi_2\sin\varphi_1 & \cos\varphi_3\cos\varphi_2 \end{bmatrix},$$

where φ_1 is the first rotation angle, φ_2 the second rotation angle, φ_3 the third rotation angle.

Rescaling. This step rescales the coordinate system $X\,Y\,Z$ and reduces the ellipsoid into a unit sphere in the final coordinate system $J\,K\,L$. Lets define the radii of the ellipsoid variogram structure along major, second-major, and minor axes as a_1, a_2, a_3 respectively. The rescaling is simply done by multiplying the following scaling matrix:

$$T = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & \frac{1}{a_3} \end{bmatrix}. \tag{1.5}$$

The transformation matrix, which transforms the original coordinate system into the final coordinate system, is: $A_{JKL \leftarrow XYZ} = TR$.

2. An Example: Data from Coastal Zone of Baltic Sea

Salinity data, which was collected in coastal zone of Baltic Sea, was used in this article. All computations have been done with package *Gstat*, which is part of R – language and environment for statistical computing and graphics. Package *Gstat* is designed for multivariable geostatistical modelling, prediction and simulation. R is free software, so it is available for wide audience.

We used some functions for generating variogram model, for calculating sample or residual variogram, for plotting a sample variogram, for fitting a variogram model to a sample variogram. For example, function

$$\texttt{Vgm}(\texttt{sill}, \texttt{model}, \texttt{range}, \texttt{nugget}, \texttt{anis})$$

generates variogram model. Argument `Anis` is anisotropy parameter, which defines ratio between the minor range and the major range and it is called anisotropy ratio. Function

$$\texttt{Variogram}(\texttt{y}, \texttt{locations}, \texttt{X}, \texttt{cutoff}, \texttt{width}, \texttt{alpha}, \texttt{beta}, \texttt{tol.hor}, \texttt{tol.ver})$$

allows to define direction in plane (x, y) (argument `alpha`), direction in z (argument `beta`), horizontal and vertical tolerance angles (arguments `tol.hor`, `tol.ver`).

First of all we detected axes of anisotropy and decided what type of anisotropy we have. We calculated variograms for many directions (with tolerance angle), fitted parametric anisotropic spherical model to a pointwise nonparametric semivariogram estimator:

$$\gamma(|h|) = \frac{1}{2N(|h|)}(z(s) - z(s+h))^2 \tag{2.1}$$

and were looking for differences between sills and ranges. In our investigation we found out that sills are nearly the same, but ranges differs a lot (see Table 1). Largest range (13786,61 m.) was obtained in Northeast direction and it represents the major axis of ellipse, the shortest range (2097,15 m.) was obtained in perpendicular direction, in Southeast and it represents the minor axis of ellipse. So we come to a conclusion, that in this case we have geometric anisotropy.

Table 1. Parameters of directional semivariograms.

Direction(alpha)	Sill	Range
North(0^o)	0.081	2097,15
NorthEast(45^o)	0.083	13786,61
East(90^o)	0.085	5663,74
SouthEast(135^o)	0.082	2097,15

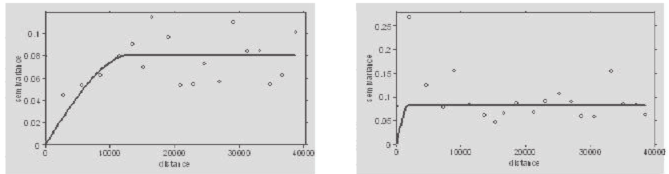
After fitting semivariogram equation in major axis direction (Northeast) is given as

$$\gamma(|h|) = 0,08\left(\frac{3|h|}{2 \cdot 13786,61} - \frac{1}{2}\left(\frac{|h|}{13786,61}\right)^3\right). \tag{2.2}$$

Semivariogram equation in minor axis direction (Southeast) is

$$\gamma(|h|) = 0,08\left(\frac{3|h|}{2\cdot 2097,15} - \frac{1}{2}\left(\frac{|h|}{2097,15}\right)^3\right). \tag{2.3}$$

Graphical representations of these functions are presented in Figure 4 a,b.



a) b)

Figure 4. Graphics of semivariograms: a) Northeast direction; b) Southeast direction.

General model of semivariogram with geometric anisotropy can be written in the following form:

$$\gamma(|h|) = \gamma(r,\beta) = \Theta_1\left(\frac{3|h|}{2\Theta_2(\beta)} - \frac{1}{2}\left(\frac{|h|}{\Theta_2(\beta)}\right)^3\right), \tag{2.4}$$

where β is argument of h, r is length of h, Θ_1 is sill, which is common in case of geometric anisotropy,

$$\Theta_2(\beta) = a_x\sqrt{\frac{1}{\cos^2(\varphi-\beta)k^2 + \sin^2(\varphi-\beta)}}$$

is a range, which depends on angle β , a_x is minor axis of anisotropy ellipse, a_y is major axis of anisotropy ellipse, φ is angle from y to the principal direction, $k = \frac{a_x}{a_y} < 1$ anisotropy ratio.

Anisotropic semivariogram model in our case is given by

$$\gamma(|h|) = \gamma(r,\beta) = \Theta_1\left(\frac{3|h|}{2\Theta_2(\beta)} - \frac{1}{2}\left(\frac{|h|}{\Theta_2(\beta)}\right)^3\right),$$

where $\Theta_2(\beta) = 2097,15\sqrt{\frac{1}{\cos^2(\frac{\pi}{4}-\beta)0,15^2+\sin^2(\frac{\pi}{4}-\beta)}}$.

References

[1] G. Arbia and G. Lafratta. Anisotropic spatial sampling designs for urban population. *Applied statistics*, **51**(2), 223 – 234, 2002.

[2] L. Budrikaitė and K. Dučinskas. Forms of anisotropy for spatial variograms. *Lietuvos matematikos rinkinys*, **44**, 542 – 546, 2004.

[3] K. Dučinskas and J. Šaltytė Benth. *Erdvinė Statistika*. KU, Klaipėda, 2003.

[4] M.D. Ecker and A.E. Gelfand. Spatial modelling and prediction under stationary non-geometric range anisotropy. *Environmental and Ecological Statistics*, 10, 165 – 178, 2003.