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ON SPLINES IN CONVEX SETS UNDER CONSTRAINTS OF TWO-SIDED INEQUALITY TYPE IN A HYPERPLANE

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Abstract. The problem of minimization of a smoothing functional under inequality constraints is considered in a hyperplane. The conditions of existence and the characteristics of a solution of this problem are obtained. It is proved that this solution is a spline. The method for its construction is suggested.

Key words: smoothing problem, spline, simplex method

1. Smoothing Histosplines

Let a mesh $\triangle_n : a = t_0 < t_1 < ... < t_n = b$ be given for the interval $[a, b]$, and let $F = \{f_1, \dots, f_n\}$ be a corresponding histogram, i.e. f_i is the frequency for the interval $[t_{i-1}, t_i]$, where $i = 1, \dots, n$. The mesh sizes are denoted by $h_i = t_i - t_{i-1}$, $i = 1, \dots, n$.

In many practical applications it is of interest to have a function g that satisfies the area matching histopolation conditions

$$\int_{t_{i-1}}^{t_i} g(t)\,dt = f_i h_i, \quad i = 1, \dots, n. \tag{1.1}$$

We will take into account that the information on the frequencies f_i, $i = 1, \dots, n$, is obtained in practice as a result of measuring, experiment or preliminary calculations and it may be inexact. Hence for given numbers $\varepsilon_i \geq 0$, $i = 1, \dots, n$, we consider more general histopolation conditions

$$\Big| \int_{t_{i-1}}^{t_i} g(t)\,dt - f_i h_i \Big| \leq \varepsilon_i, \quad i = 1, \dots, n, \tag{1.2}$$

and pose the following problem.

Problem 1

$$\int_a^b (g^{(q)}(t))^2 \, dt \longrightarrow \min_{g\in D_1(\varepsilon)},$$

$$D_1(\varepsilon) = \{g: \quad g \in W_2^q[a,b], \quad \Big| \int_{t_{i-1}}^{t_i} g(t)dt - f_i h_i \Big| \le \varepsilon_i, \; i = 1,\ldots,n\},$$

where $W_2^q[a,b]$ is the Sobolev space.

In the case of exact information (i.e. $\varepsilon_i = 0$ for all i) we have a histopolation problem the solution of which is a spline s (called *a histospline*) from the space $S(\triangle_n)$ of integral splines of degree $2q$ and defect 1 over the mesh $\triangle_n$ (e.g. [5]):

$$S_{2q,1}(\triangle_n) = \{s \in W_2^q[a,b] \; : \; \int_{t_{i-1}}^{t_i} g(t) \; dt = 0, \; i = 1,\ldots,n,$$
$$\Longrightarrow \int_a^b g^{(q)}(t) s^{(q)}(t) \; dt = 0 \text{ for all } g \in W_2^q[a,b] \; \}.$$

In the case of inexact information (i.e. $\varepsilon_i > 0$ for some i) it is a problem of smoothing histopolation. If $n \le q$, then any polynomial of degree $q-1$, which satisfies the condition of histopolation (1.2), gives the solution of Problem 1. If $n > q$ and no algebraic polynomial of degree $q-1$ satisfies the inequalities (1.2), then Problem 1 has a unique solution (e.g. [5]). This solution is a spline from the space $S_{2q,1}(\triangle_n)$, which minimizes the smoothing functional under restrictions. This spline is called *a smoothing histospline*.

The main purpose of the present paper is to consider Problem 1 with one additional restriction. We pose the following problem.

Problem 2

$$\int_a^b (g^{(q)}(t))^2 \, dt \longrightarrow \min_{g\in D_2(\varepsilon)},$$

$$D_2(\varepsilon) = \{g: \quad g \in W_2^q[a,b], \quad \Big| \int_{t_{i-1}}^{t_i} g(t)\, dt - f_i h_i \Big| \le \varepsilon_i, \; i = 1,\ldots,n,$$
$$\int_a^b g(t)\, dt = 1\},$$

which naturally appears under approximation of a given histogram F with frequencies f_i ,$i = 1,\ldots,n$.

We investigate this problem in a more general case in a Hilbert space (see Problem 3) and obtain the existence and the characteristics of its solution.

We reduce Problem 3 to the problem of "almost" linear programming with some nonlinear conditions (see Problem 5) and suggest the method for finding its solution by the modification of the simplex algorithm.

2. The Generalization of the Problem of Smoothing Histopolation

Let X, Y be Hilbert spaces and assume that a linear operator $T : X \to Y$ and functionals $k_i : X \to \mathbb{R}, i = 1, \ldots, n$, are continuous. For given vectors $\boldsymbol{r} = (r_1, \ldots, r_n)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i \geq 0,\ i = 1, \ldots, n$, we consider the conditional minimization problem.

Problem 3

$$\| Tx \|_Y \longrightarrow \min_{\substack{|k_i x - r_i| \leq \varepsilon_i,\ i=1,\ldots,n, \\ \sum_{i=1}^n k_i x = \sum_{i=1}^n r_i.}}$$

In the case $\varepsilon_i = 0,\ i = 1,\ \ldots\ , n,$ a solution of this problem is called an interpolating spline for a vector $\boldsymbol{r}$ and it belongs to the space

$$S(T, A) = \{s \in X\colon \ \ < Ts, Tx >= 0 \text{ for all } \ x \in \mathrm{Ker} A\},$$

corresponding to the operators T and $A = (k_1, \ldots, k_n)$. In the case of inexact information ($\varepsilon_i > 0$ for some i) Problem 3 without the last condition defines *splines in a convex set* (in the special case *smoothing splines*) [5, 6]. Such splines belong to the space $S(T, A)$ also.

Let us suppose that $\mathrm{Im} A = \mathbb{R}^n,\ \mathrm{Im} T = Y$ and the sum $\mathrm{Ker} T + \mathrm{Ker} A$ is closed. Let us denote

$$Z_{\boldsymbol{r}} = \{z \in \mathbb{R}^n\colon \sum_{i=1}^n z_i = \sum_{i=1}^n r_i\}, \quad X_{\boldsymbol{r}} = \{x \in X\colon Ax \in Z_{\boldsymbol{r}}\},$$

$$P_{\boldsymbol{r},\boldsymbol{\varepsilon}} = \prod_{i=1}^n [r_i - \varepsilon_i; r_i + \varepsilon_i], \quad C_{\boldsymbol{r},\boldsymbol{\varepsilon}} = \{x \in X_{\boldsymbol{r}}\colon Ax \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}}\}.$$

We rewrite Problem 3 in the form

$$||Tx||_Y \longrightarrow \min_{x \in C_{r,\varepsilon}}$$

and prove the following results.

Theorem 1. *A solution of Problem 3 exists. An element $\sigma \in C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ is a solution of this problem if and only if there exists an element $\boldsymbol{\lambda} \in \mathbb{R}^n$ such that*

$$T^*T(\sigma) = A^*\boldsymbol{\lambda} \ \textit{and} \ < \boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma > \geq 0 \ \ \textit{for all } \boldsymbol{\omega} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}.$$

Corollary 1. A solution of Problem 3 belongs to the space $S(T,A)$ of splines.

Theorem 2. *An element $\sigma \in C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ is a solution of Problem 3 if and only if there exist elements $\boldsymbol{\lambda} \in I\!R^n$ and $\gamma \in I\!R$ such that*

$$T^*T(\sigma) = A^*\boldsymbol{\lambda},$$

$$\begin{array}{lll} \lambda_i = \gamma & \text{if} & |k_i\sigma - r_i| < \varepsilon_i, \\ \lambda_i \geq \gamma & \text{if} & k_i\sigma - r_i = -\varepsilon_i, \\ \lambda_i \leq \gamma & \text{if} & k_i\sigma - r_i = \varepsilon_i, \quad \text{for} \quad i = 1,\dots,n. \end{array}$$

3. The Equivalent Problem of Quadratic Programming

Taking into account that the solution of Problem 3 is a spline, we can restrict the class of functions X by the space $S(T,A)$ and rewrite the smoothing functional $||Tx||_Y$ as a function of n new non-negative variables

$$z_i = k_i s - r_i + \varepsilon_i, \quad i = 1,\dots,n. \tag{3.1}$$

If we denote by $s_i \in S(T,A)$ the spline which satisfies the conditions

$$k_j s_i = \delta_{ij}, \quad j = 1,\dots,n, \quad i = 1,\dots,n,$$

where δ_{ij} is the Kronecker symbol, then $s_1,\dots,s_n$ is a basis of the space $S(T,A)$. By introducing the matrix $\mathbf{D} = (\lambda_{ji})_{i,j=1,\dots,n}$, where $(\lambda_{ij})_{j=1,\dots,n}$ are the coefficients of the basis spline s_i, and the vectors $\boldsymbol{z} = (z_i)_{i=1,\dots,n}$, and $\boldsymbol{c} = (c_i)_{i=1,\dots,n}$, where $c_i = \sum_{j=1}^{n}(r_j - \varepsilon_j)(\lambda_{ji} + \lambda_{ij})$, Problem 3 can be rewritten in the matrix form

Problem 4

$$\boldsymbol{z}\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}\boldsymbol{z}^T \longrightarrow \min_{\boldsymbol{z} \geq \boldsymbol{\theta}, \quad \boldsymbol{z} \leq 2\boldsymbol{\varepsilon}, \quad (\boldsymbol{z}-\boldsymbol{\varepsilon})\boldsymbol{e}^T = 0,}$$

where $\boldsymbol{e}$ and $\boldsymbol{\theta}$ are the vectors with n unit components and n zero components correspondingly.

Lemma 1. *The matrix $\mathbf{D}$ is symmetric and positive semidefinite.*

Thus Problem 3 is reduced to Problem 4 of quadratic programming with symmetric and positive semidefinite matrix under linear restrictions.

4. The Equivalent Problem of "Almost" Linear Programming under Nonlinear Conditions

We use the Wolfe method (e.g. [4]) to reduce Problem 4 to the problem of "almost" linear programming with some nonlinear conditions. The reasoning in this reduction is similar to that of [1, 2] and we consider only important steps.

We start with the Lagrange function

$$F(\boldsymbol{z}, \boldsymbol{\lambda}) = \boldsymbol{z}\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}\boldsymbol{z}^T + \boldsymbol{\lambda}^1(\boldsymbol{z} - 2\boldsymbol{\varepsilon})^T + \lambda^0(\boldsymbol{z} - \boldsymbol{\varepsilon})\boldsymbol{e}^T,$$

where $\boldsymbol{\lambda} = (\lambda^0, \boldsymbol{\lambda}^1)$ is the vector of Lagrange multipliers,

$$\lambda^0 \in I\!R,\ \boldsymbol{\lambda}^1 = \ (\lambda_i)_{i=1,\ldots,n} \in I\!R^n.$$

Taking into account necessary and sufficient conditions for $\boldsymbol{z}$ to be a solution of Problem 4 (e.g. [4]) by introducing slack non-negative variables ($\bar{\boldsymbol{z}} = (\bar{z}_i)_{i=1,\ldots,n}$ and $\boldsymbol{\mu} = (\mu_i)_{i=1,\ldots,n}$ as $\boldsymbol{\mu}^T = 2(\mathbf{D}\boldsymbol{z}^T) + \boldsymbol{c}^T + (\boldsymbol{\lambda}^1)^T + \lambda^0\boldsymbol{e}^T$ and $\bar{\boldsymbol{z}} = 2\boldsymbol{\varepsilon} - \boldsymbol{z}$, we can rewrite Problem 4 as a linear programming minimization problem of $\boldsymbol{u}\boldsymbol{e}^T$ for an auxiliary non-negative vector $\boldsymbol{u} = (u_i)_{i=1,\ldots,n}$ under some nonlinear restrictions.

Problem 5

$$\boldsymbol{u}\boldsymbol{e}^T \longrightarrow \min$$

$$\begin{aligned}
&2\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}^T + (\boldsymbol{\lambda}^1)^T + \lambda^0\boldsymbol{e}^T - \boldsymbol{\mu}^T + \mathbf{E}\boldsymbol{u}^T = 0,\\
&\boldsymbol{z} + \bar{\boldsymbol{z}} = 2\boldsymbol{\varepsilon},\ (\boldsymbol{z} - \boldsymbol{\varepsilon})\boldsymbol{e}^T = 0,\\
&\boldsymbol{\mu}\boldsymbol{z}^T = 0,\ \boldsymbol{\lambda}^1\bar{\boldsymbol{z}}^T = 0,\\
&\boldsymbol{z} \geq \boldsymbol{\theta},\ \bar{\boldsymbol{z}} \geq \boldsymbol{\theta},\ \boldsymbol{\lambda}^1 \geq \boldsymbol{\theta}, \boldsymbol{\mu} \geq \boldsymbol{\theta},\ \boldsymbol{u} \geq \boldsymbol{\theta},
\end{aligned}$$

where $\mathbf{E}$ is the diagonal matrix with components 0, 1 and -1. The existence of a non-negative solution of Problem 3 implies that zero is the solution of Problem 5.

Theorem 3. *Let Problem 3 have the unique solution. Then it is equivalent to Problem 5 in the following sense*

- *Problem 5 has the unique solution too.*
- *The solution of Problem 3 determines the solution of Problem 5 and the solution of Problem 5 determines the solution of Problem 3 by* (3.1).

5. The Modification of the Simplex Method

Problem 5 differs from problems of linear programming in two simple nonlinear conditions

$$\boldsymbol{\mu}\boldsymbol{z}^\top = 0, \quad \boldsymbol{\lambda}(\bar{\boldsymbol{z}})^\top = 0.$$

For the solution of the problem a modification of the simplex method based on the Wolfe and Daugavet works ([3, 4]) is suggested. We give a short description of this algorithm.

Initial plan

We choose $\boldsymbol{z} = \bar{\boldsymbol{z}} = \boldsymbol{\varepsilon}$, $\boldsymbol{\lambda} = \boldsymbol{\theta}$, $\boldsymbol{\mu} = \boldsymbol{\theta}$. We take an initial value of

$$u_i = |2(\mathbf{D}\boldsymbol{z}^T)_i + c_i|, \quad i = 1, \ldots, n,$$

and choose the sign at u_i $i = 1, \ldots, n$ (the diagonal elements of matrix $\mathbf{E}$) in such a way that they satisfy the equations

$$2\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}^T + \mathbf{E}\boldsymbol{u}^T = 0.$$

Iterations

Every step of the method is a transformation of the simplex table, taking into account the lexicographic ordering (it allows us to avoid iterative loops) and the additional conditions $\boldsymbol{\mu}\boldsymbol{z}^\top = 0$, $\boldsymbol{\lambda}^1\bar{\boldsymbol{z}}^T = 0$. We can show that the additional nonlinear condition does not prevent us from doing it. We prove that if the next simplex iteration can not be done without violation of these nonlinear conditions then the last basic solution gives $\boldsymbol{u}\boldsymbol{e}^T$=0, i.e. the solution of Problem 5.

Solution

This method gives us the values of the components of the vector $(\boldsymbol{r}-\boldsymbol{\varepsilon}+\boldsymbol{z})$. The corresponding interpolating spline is the solution of Problem 3. It can be constructed by some known methods of construction of interpolating splines.

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