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RELAXATION METHOD FOR ELLIPTIC PROBLEM WITH LOGARITHMIC CONVERGENCE

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Abstract. In this paper new iterative relaxation method for solution of elliptic problems is proposed. The method converges with logarithmic rate and outperforms many known iterative methods. It was successfully applied to one-, two- and three-dimensional elliptic problems.

Key words: elliptic PDE, iterative relaxation method, optimal set of steps

1. Introduction

Let us consider elliptic PDE with R – dimensional positively definite operator, which is the sum of one-dimensional commutative operators:

$$L\,(\mathrm{x})\,u + f\,(x) = 0, \qquad L\,(\mathrm{x}) = \sum_{r=1}^{R} L_r\,(x_r). \tag{1.1}$$

Let us introduce multidimensional orthogonal grid and write a finite-difference approximation of this equation:

$$\Lambda u \equiv \sum_{r=1}^{R} \Lambda_r u = f \tag{1.2}$$

If $R \geq 2$ then it is difficult to solve this equation by direct methods. The most reliable iterative methods are the explicit iterative algorithm with Chebyshev's set of parameters and Conjugate Gradient method [3]. But for these methods (when preconditioning matrix is not used) the number of iterations

$$K \sim \mu = \sqrt{\lambda_{\max}/\lambda_{\min}},$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the bounds of the spectrum of matrix Λ. For simple elliptic PDE the stiffness number $\lambda_{\max}/\lambda_{\min} \sim N_r^2$, where N_r is a number of grid points per one space direction. So applied numerical problems require $K \approx 1000$ iterations. It is too large computational cost for many real world applications.

In this paper we introduce a new iterative method, which applies the relaxation method for the implicit time-factorized scheme. It is proved that for this method the number of iterations $K \sim \ln(\lambda_{\max}/\lambda_{\min})$. Thus it is one of fastest known iterative method, comparable with multigrid methods.

2. Relaxation Method

Elliptic PDE (1.1) is often solved with the relaxation method [3]. Let us consider the evolutionary problem with the same discrete spatial operator:

$$\frac{du}{dt} = \Lambda u + f, \quad \Lambda = \sum_{r=1}^{R} \Lambda_r, \tag{2.1}$$

a stationary limit of which gives the solution of (1.2). For solving the system of ODEs (2.1) we use the following time-factorized scheme [1]:

$$\prod_{r=1}^{R} \left(E - \frac{\tau}{2}\Lambda_r\right) \frac{\hat{u} - u}{\tau} = \Lambda u - f. \tag{2.2}$$

It is simple, reliable and in the case of two spatial dimensions it is identical to the Peaceman-Rachford scheme [2]. The number of iterations which is required to solve the stationary problem (1.2) with the time-factorized scheme depends on the set of time steps (or iterative parameters) used in computations.

3. The Optimal Step

The stability functions of the time-factorized scheme (2.2) in one- and two-dimensions are defined as follow:

$$\rho = \frac{1 - \frac{\tau\lambda}{2}}{1 + \frac{\tau\lambda}{2}}, \quad R = 1; \quad \rho = \frac{1 - \frac{\tau\lambda_1}{2}}{1 + \frac{\tau\lambda_1}{2}} \frac{1 - \frac{\tau\lambda_2}{2}}{1 + \frac{\tau\lambda_2}{2}}, \quad R = 2. \tag{3.1}$$

They are monotonously decreasing functions of the argument $\tau\lambda$. Equating its value for the lowest harmonic $\rho(\tau\lambda_m)$ to the value of the highest harmonic $\rho(\tau\lambda_M)$ we obtain the optimal constant step (the iterative parameter), which leads to the minimal number of iterations.

Theorem 1. *In 1D case the optimal constant time step is defined by* $\tau_0 = \frac{2}{\sqrt{\lambda_m \lambda_M}}$ *and the minimal number of iterations is* $K(\tau_0) = \frac{1}{2}\sqrt{\frac{\lambda_M}{\lambda_m}} \ln \frac{1}{\varepsilon}$ *, where* λ_m, λ_M *are bounds of the spectrum of operator* Λ.

We note that one-dimensional case and Theorem 1 are considered only as simple examples, since such problems can be solved efficiently by direct methods. Similarly to one-dimensional case we can obtain the optimal constant time step and corresponding number of iterations sufficient to decrease the error of each component of the error function. The decomposition is done by using eigenvectors of operator Λ. In two-dimensional case the following theorem is valid:

Theorem 2. *For two-dimensional problems the optimal time step is defined by*

$$\tau_0 = 2\sqrt{\frac{\lambda_{1M}+\lambda_{2M}-\lambda_{1m}-\lambda_{2m}}{\lambda_{1m}\lambda_{1M}\left(\lambda_{2M}-\lambda_{2m}\right)+\lambda_{2m}\lambda_{2M}\left(\lambda_{1M}-\lambda_{1m}\right)}}$$

$$\approx 2\sqrt{\frac{\lambda_{1M}+\lambda_{2M}}{\lambda_{1M}\lambda_{2M}\left(\lambda_{1m}+\lambda_{2m}\right)}}$$

and the number of iterations is given by

$$K\left(\tau_0\right) = \ln\frac{1}{\varepsilon}\,\ln^{-1}\left(\frac{1+\frac{1}{2}\tau_0\left(\lambda_{1m}+\lambda_{2m}\right)}{1-\frac{1}{2}\tau_0\left(\lambda_{1m}+\lambda_{2m}\right)}\right).$$

Here we have supposed that $\lambda_{1M}, \lambda_{2M} \gg \lambda_{1m}, \lambda_{1m}$.

4. Quasi-Spectral Set of Steps

A constant optimal step used at each iteration does not allow to construct the fast iterative method. So we need to use a set of such steps. For example using the Chebyshev iterative parameters for the explicit iterative algorithm one obtains the same accuracy with much smaller number of iterations. In one-dimensional case the following theorem gives the set of steps for the time-factorized scheme, which guarantees the logarithmic convergence speed.

Theorem 3. *Let decompose the error function in the basis of eigenvectors of operator Λ. In order to decrease every component of error at least $1/\varepsilon$ times it is sufficient to do*

$$K \geq \ln\frac{1}{\varepsilon}\ln\sqrt{\frac{\lambda_M}{\lambda_{\mathrm{m}}}} \tag{4.1}$$

iterations and to use the following set of iterative parameters $\{\tau_k, 0 \leq k \leq K\}$

$$\tau_k = \frac{2}{\lambda_m}\left(\frac{\lambda_m}{\lambda_M}\right)^{\frac{k}{K}}, 0 \leq k \leq K \tag{4.2}$$

In applied computations ε has the same order as the accuracy of solving the linear system of equations. In R-dimensional case

$$\lambda_m = \min_r \lambda_{r\,\min}, \quad \lambda_M = \max_r \lambda_{r\,\max}.$$

The following statement holds.

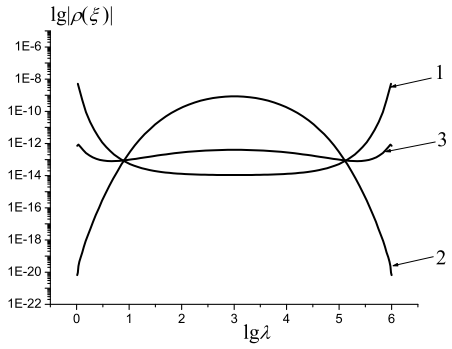

Figure 1. Envelop of the extremum absolute values of the stability function for the time-factorized scheme (2.2).

Theorem 4. *In two-dimensional case the quasi-spectral set of steps* (4.2) *also provides logarithmic convergence rate and it is sufficient to do only half number of iterations defined by* (4.1).

The proof of both theorems is based on the representation of the stability function after K steps in the form

$$\rho_K(\xi) = \prod_{k=1}^{K} \operatorname{th}(\xi - \eta_k), \quad \xi \in \left[\ln\sqrt{\lambda_m}, \ln\sqrt{\lambda_M}\right]$$

and the inequality $thx < e^{-2e^{-2x}}$. This inequality gives us a possibility to estimate the stability function and at the same time to replace product $\prod_{k=1}^{K} \operatorname{th}(\xi - \eta_k)$ by geometric series. For three-dimensional case there is no such strict theoretical result, but practical calculations show that the convergence rate remains logarithmic. This is due to the fact that stability function is much more complex in three-dimensional case and it explains why the theoretical line in Fig. 1 lies below the line defined by computational examples. Also we tried to improve the method and studied other sets of steps. Fig. 1 illustrates the envelop of the extremum absolute values of the stability function for the time-factorized scheme (2.2).

The curve 1 corresponds to the quasi-spectral set of steps (4.2), curve 2 – to the quasi-Chebyshev set of iterative parameters

$$\tau_k = \frac{2}{\sqrt{\lambda_m \lambda_M}} \left(\frac{\lambda_m}{\lambda_M}\right)^{\frac{1}{2}\cos\frac{\pi(k-1/2)}{K}}, \quad 1 \le k \le K. \tag{4.3}$$

Slightly better results were obtained for the interpolation set of steps (see curve 3)

$$\tau_k = \frac{2}{\lambda_m}\left(\frac{\lambda_m}{\lambda_M}\right)^{\frac{1+z_k}{2}}, \quad z_k = \theta_k\left(1+\left(1-\theta_k^2\right)/(2b)\right)^b,$$

$$b = \left(1+\frac{1}{8}\ln^2\left(\frac{\lambda_M}{\lambda_m}\right)\right)^{-1}, \quad \theta_k = \frac{2k}{K}-1, \quad 0 \le k \le K.$$

but the performance gain is insignificant.

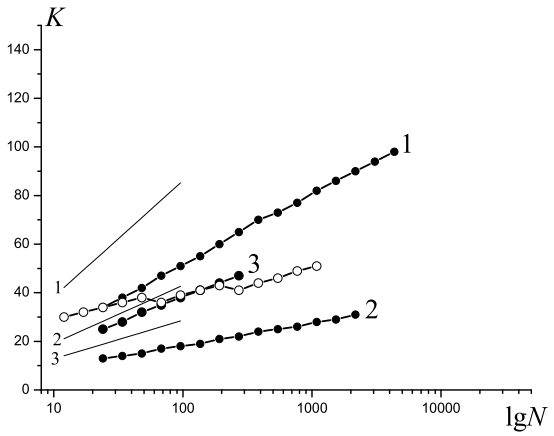

Figure 2. Number of steps K depending on the grid nodes per one space direction in logarithmic scale.

5. Practical Computations

This method was successfully applied to solve the typical 1,2,3-dimensional heat conduction problems, which are typical examples of elliptic problems. Fig. 2 illustrates the results of these computations. Curves 1,2,3 correspond to the 1,2,3-dimensional cases accordingly. Thin lines represent the theoretical estimations. Line with circles represents the computations in the two-dimensional unbounded area. One can see that calculations in two- and three-dimensional cases require less iterations than in one-dimension case. In all cases the number of iterations $K \sim \lg N$ therefore $K \sim \lg \sqrt{\lambda_M/\lambda_m}$.

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