

# ON THE COMPUTATION OF THE PROBABILITY DENSITY FUNCTION OF $\alpha$ -STABLE DISTRIBUTIONS

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**Abstract.** Stable laws have a wide sphere of application: probability theory, physics, electronics, economics, sociology. They also play an important role in financial mathematics, since the classical models of financial market based on the hypothesis of normality often become inadequate. However, the practical implementation of stable models is a nontrivial task, because (with few exceptions) probability density functions of  $\alpha$ -stable distributions have no analytical representation. In our previous works Zolotarev representation of the probability density function and direct numerical integration method (with 96-point Gaussian quadrature) were used, but it seems that the application of another probability density function integral representation (with Gaussian quadrature and Laguerre quadrature combination) is more efficient in computational aspect, fairly precise and easily implementable. The suggested method is compared with other algorithms.

**Key words:**  $\alpha$ -stable distributions, density approximation, Laguerre quadrature, financial modelling

## 1. Introduction

The stable distributions have a wide area of applications: probability theory, communications theory, physics, astronomy, economics, sociology. The symmetric stable distributions were introduced by Cauchy in 1853, and one of them is known by his name. The notion of stable probability law was introduced by Lévy in 1924. The class originated from the desire to generalize the Central Limit Theorem. It can be shown, that all limit distributions of sums of i. i. d. random variables must be stable [5].

All but one of these distributions have infinite variance. The infinite variance of these random variables implies that observations of very large magnitude can be expected and may, in fact, dominate sums of these random

variables. It is not valid to treat these observations as outliers since excluding them takes away much of the significance of the original data; indeed, it is precisely these observations that may be of greatest interest. This led Mandelbrot [7, 8] to suggest the stable laws as possible models for the distribution of income and speculative prices. Take for example the distribution of changes in stock market prices. Mandelbrot [8], Fama [4] and others have shown that the probability of very large deviations is so great, that many statistical techniques which depend for their validity on the asymptotic theory of finite variance distributions are inapplicable. The sum of a large number of these variables is often dominated by one of the summands - a theoretical property of infinite variance distributions. In such a case, a mathematical model assuming such a distribution for the observations is very useful: financial data are typically asymmetric, peaked and heavy-tailed [13], and so it better (than usual normal models) fits the empirical data distribution.

## 2. $\alpha$ -Stable Distributions

The stable distribution can be most conveniently described by its characteristic function

$$\log \varphi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ -\sigma |t| \{1 + i\beta \operatorname{sign}(t) \frac{2}{\pi} \log |t|\} + i\mu t, & \alpha = 1, \end{cases} \quad (2.1)$$

where  $\alpha \in (0, 2]$ ,  $\beta \in [-1, 1]$ ,  $\sigma > 0$ ,  $\mu \in \mathbb{R}$ . Since (2.1) is characterized by four parameters we will denote  $\alpha$ -stable distributions by  $S_\alpha(\sigma, \beta, \mu)$ . Here  $\alpha$  is the characteristic exponent (index),  $\sigma$  is the scale parameter,  $\beta$  is the skewness, and  $\mu$  is the location parameter. A stable probability density function (PDF) is symmetrical if and only if  $\beta = 0$ . When  $\sigma = 1$  and  $\mu = 0$  the distribution is called standard stable. The general PDF of the stable distribution can be standardized such that

$$p(x, \alpha, \beta, \mu, \sigma) = \frac{1}{\sigma} p\left(\frac{x - \mu}{\sigma}, \alpha, \beta, 0, 1\right), \quad (2.2)$$

$$p(x, \alpha, \beta) = \frac{1}{\pi} \int_0^\infty \exp(-t^\alpha) \cos(tx - \beta t^\alpha \tan \frac{\pi\alpha}{2}) dt, \quad \alpha \neq 1. \quad (2.3)$$

The canonical representation (2.1) has one serious disadvantage. The functions  $\varphi(t)$  have discontinuities at all points of the form  $a = 1, \beta \neq 0$ . Therefore for numerical purposes it is advisable to use Nolan's [11] parametrization

$$\log \varphi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha (1 + i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2} ((\sigma|t|)^{1-\alpha} - 1)) + i\mu_0 t, & \alpha \neq 1, \\ \sigma |t| (1 + i\beta \operatorname{sign}(t) \frac{2}{\pi} \log(\sigma|t|)) + i\mu_0 t, & \alpha = 1. \end{cases} \quad (2.4)$$

The  $S_\alpha^0(\sigma, \beta, \mu_0)$  parametrization is a variant of Zolotariov's ( $M$ ) parametrization [15], with the characteristic function and hence the density and the distribution function jointly continuous in all four parameters. The location parameters of the two representations are related by

$$\mu_0 = \begin{cases} \mu + \beta\sigma \tan \frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \mu + \beta\sigma \frac{2}{\pi} \ln \sigma, & \alpha = 1. \end{cases}$$

The PDF of stable random variables exist and are continuous but, with a few exceptions they are not known in closed form. The exceptions are

1. Gaussian distribution:  $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$ .
2. Cauchy distribution:  $S_1(\sigma, 0, \mu)$ .
3. Lévy distributions:  $S_{0.5}(\sigma, \pm 1, \mu)$ .

### 3. Methods of Evaluation of the PDF of Stable Distributions

With no analytical representation of probability density functions the practical implementation of stable models is a nontrivial task. However, there are numerical methods that have been found useful in practice and are discussed in this section.

#### 3.1. Direct numerical integration methods

DuMouchel [3] developed a procedure for approximating the cumulative distribution function of  $S_\alpha$  by using Bergström's [1] series expansion and Zolotarev's [15] representation. Holt and Crow [6] combined four alternative procedures (depending on the particular range of  $\alpha$  and  $\beta$ ) to approximate an inversion integral for computing PDF values from the characteristic function. Both DuMouchel and Holt-Crow algorithms are computationally intensive and time consuming, making, for example, maximum likelihood estimation (of parameters) a nontrivial task.

Nolan [12] used a variant of Zolotarev's ( $M$ ) parametrization (2.4) and computed the density by numerically evaluating the integral

$$p(x, \alpha, \beta) = \frac{1}{\pi} \int_0^\infty \cos(h(x, t; \alpha, \beta)) \exp(-t^\alpha) dt,$$

where

$$h(x, t; \alpha, \beta) = \begin{cases} xt + \beta \tan \frac{\pi\alpha}{2} (t - t^\alpha), & \alpha \neq 1, \\ \mu + \beta\sigma \frac{2}{\pi} \ln \sigma, & \alpha = 1 \end{cases}$$

is the obtained form of the inversion formula. He splits the region of integration into intervals where the cosine term change sign. The endpoints of the intervals are found analytically when  $\beta = 0$ ; in other cases the endpoints are found numerically, and the integral is approximated on each interval. Nolan reports, that then  $\alpha < 1$  the exponent  $\exp(-t^\alpha)$  decreases slowly and the region of integration must get larger to obtain sufficient accuracy, computations are slower. When  $\alpha < 1$ , there are usually too many subintervals, round-off errors increase quickly, and the desired accuracy (six digits) cannot be achieved. Besides, when  $\beta \neq 0$  and  $0 < |\alpha - 1| < 0,001$ , there are numerical problems in

computing the term  $\tan \frac{\pi\alpha}{2}(t - t^\alpha)$ . Moreover, when  $x$  is large, the integrand oscillates very fast.

One more direct numerical integration method should be mentioned. In the case of  $\alpha > 1$  we replace the PDF integral with the approximation

$$p(x, \alpha, \beta) \approx \int_0^\Delta f(x, t; \alpha, \beta) dt, \quad (3.1)$$

where  $\Delta = \Delta(\alpha, \varepsilon)$  is the root of  $\Gamma(\frac{1}{\alpha}, \Delta^\alpha) = \alpha\pi\varepsilon$  equation (the error of (3.1) is non greater than  $\varepsilon$ ) and we evaluate (3.1) via some adaptive integration method with an accuracy  $\delta$ . This approach is rather effective in computational aspect if we apply 96-points Gaussian quadratures and  $\delta < 10^{-10}$ .

### 3.2. Fast Fourier Transform method

Chenyao, Mittnik and Doganoglu [10] presented an algorithm for calculating the PDF of stable distribution by employing the Fast Fourier Transform (FFT). In terms of the characteristic function the PDF can be written as

$$p(x, \alpha, \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt. \quad (3.2)$$

The integral is calculated for  $N$  equally-spaced points with distance  $h$ , namely  $x_k = (k - 1 - \frac{N}{2})h$ ,  $k = 1, \dots, N$ . Letting  $t = 2\pi\omega$ , (3.2) becomes

$$p((k - 1 - \frac{N}{2})h) = \int_{-\infty}^{\infty} \varphi(2\pi\omega) e^{-i2\pi\omega(k-1-\frac{N}{2})h} d\omega. \quad (3.3)$$

The integral in (3.3) can be approximated by using the rectangle rule for  $N$  points with spacing  $s$ , i.e.,

$$p((k - 1 - \frac{N}{2})h) \approx s \sum_{n=1}^N \varphi(2\pi s(n - 1 - \frac{N}{2})) e^{-i2\pi(n-1-\frac{N}{2})(k-1-\frac{N}{2})hs}. \quad (3.4)$$

By setting in (3.4)  $s = \frac{1}{hN}$ , one obtains approximation

$$p((k - 1 - \frac{N}{2})h) \approx s(-1)^{k-1-\frac{N}{2}} \sum_{n=1}^N (-1)^{n-1} \varphi(2\pi s(n - 1 - \frac{N}{2})) e^{\frac{-i2\pi(n-1)(k-1)}{N}}. \quad (3.5)$$

The summation in (3.5) is computed by applying FFT to the sequence

$$(-1)^{n-1} \varphi(2\pi s(n - 1 - \frac{N}{2})), \quad n = 1, \dots, N.$$

The  $k^{th}$  element of the resulting sequence is normalized by  $s(-1)^{k-1-\frac{N}{2}}$ , to obtain the PDF value for each grid point.

By substituting the standard stable characteristic function ( $\alpha \neq 1$ )

$$\varphi(t) = \exp(-|t|^\alpha + i\beta|t|^\alpha \text{sign}(t) \tan \frac{\pi\alpha}{2})$$

into (3.5), with  $t = 2\pi s(n - 1 - \frac{N}{2})$ , standardized PDF values can be calculated and, via (2.2), transformed to any desired parameter combination.

So the procedure for obtaining PDF consists of two steps. First, specify equally-spaced grid values and compute the PDF values for these grid points (to make use of the efficient FFT the numbers of grid points,  $N$ , should be a power of 2). In a second step, linear interpolation is applied to the data points falling between grid values. Authors state that comparisons with two other approaches, namely that of [6, 12], demonstrate that the proposed algorithm generates accurate PDF values (deviations can be found only in the sixth digit, in case  $\alpha \geq 1, 25$ ), if  $N \geq 2^{13}$ , and order-of-complexity calculations show that in most practical situations, such as empirical estimation, the FFT-based method is more efficient than direct numerical integration.

So, direct numerical integration methods [6, 12] call for the calculation of numerous subintegrals, FFT-based methods [9, 10] call for the employment of sophisticated subroutines.

### 3.3. The method of two quadratures

Let us take standard stable PDF, case  $\alpha \neq 1$ , in Nolan's parametrization (Fig. 1):

$$p(x, \alpha, \beta) = \frac{1}{\pi} \int_0^\infty \exp(-t^\alpha) \cos(xt + \beta \tan \frac{\pi\alpha}{2}(t - t^\alpha)) dt.$$

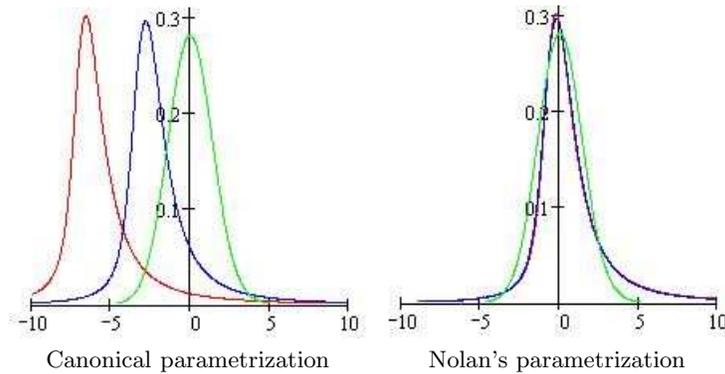
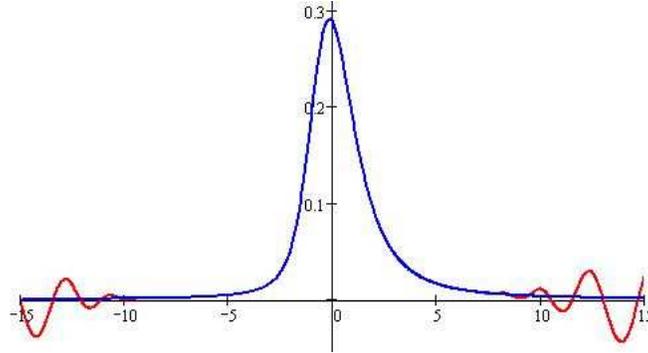


Figure 1. Stable densities with  $\beta = 0.5$ ,  $\alpha = 1.05, 1.12, 1.95$ .

The most 'natural' way of evaluating the integral over the interval  $(0, \infty)$  seems to be 96-points Laguerre quadrature

$$p(x, \alpha, \beta) \approx \frac{1}{\pi} \sum_{n=1}^{96} w_n \exp(t_n - t_n^\alpha) \cos(xt_n + \beta \tan \frac{\pi\alpha}{2}(t_n - t_n^\alpha)), \quad (3.6)$$

where the abscissas  $t_n$  are given by the roots of the Laguerre polynomial  $L_{96}(t)$ , and the weights  $w_n = t_n(97L_{97}(t_n))^{-2}$ . If direct implementation of the approximation (3.6) does not provide sufficient accuracy of PDF values (because of fast oscillation of the Laguerre quadrature when  $x$  is large (see Fig. 2)), then we can 'suppress' the oscillation by decreasing the power of



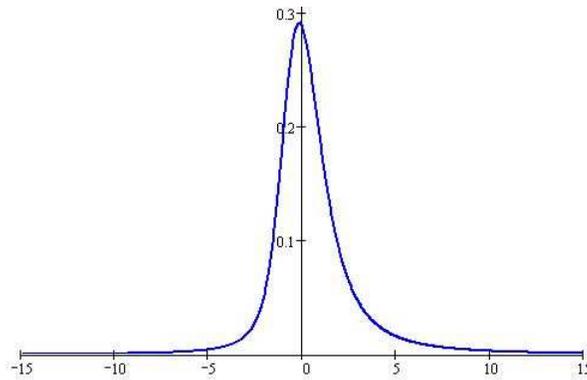
**Figure 2.** Approximation with 96-points Laguerre quadrature,  $\alpha = 1.25$ ,  $\beta = 0.5$ .

exponent in the integrand by shifting (see Fig. 3)

$$p(x, \alpha, \beta) = \int_0^d f(x, t; \alpha, \beta) dt + \int_0^\infty f(x, u + d; \alpha, \beta) du, \quad (3.7)$$

where

$$f(x, t, \alpha, \beta) = \frac{1}{\pi} \exp(-t^\alpha) \cos(xt + \beta \tan \frac{\pi\alpha}{2}(t - t^\alpha)).$$



**Figure 3.** Approximation with two quadratures,  $\alpha = 1.25$ ,  $\beta = 0.5$ .

Now the first integral we evaluate with 96-points Gaussian quadrature, and the 'tail' - with 96-points Laguerre quadrature. For large absolute  $x$ -values we will use an asymptotic Bergström [1, 2, 14] expansion (here given in canonical parametrization)

$$p(x, \alpha, \beta) = \frac{1}{\pi x} \sum_{n=1}^N a_n x^{-\alpha n} + O(x^{-(N+1)\alpha-1}),$$

$$a_n = \frac{(-1)^{n-1} \Gamma(n\alpha + 1)}{n!} \left(1 + \beta^2 \tan^2 \frac{\pi\alpha}{2}\right)^{\frac{n}{2}} \sin n \left(\frac{\pi\alpha}{2} + \arctan \left(b \tan \frac{\pi\alpha}{2}\right)\right),$$

which yields efficient approximations for tails of the density.

We compare the accuracy of this method vs the direct numerical integration method (benchmark). The scheme is analogical to the comparison in [10]. We calculate standard PDF values for  $\alpha = 1.25, 1.50, 1.75, \beta = -1.0, -0.5, 0, 0.5, 1.0$  with  $d = 8$  and  $x = 0, 0, 1, \dots, 4.9$ . We use the mean absolute deviation ( $D_1$ ) and the maximum absolute deviation ( $D_2$ ) of the vectors of the PDF values, defined by

$$D_1 = \frac{1}{N} \sum_{i=1}^N |v_i - w_i|, \quad D_2 = \max_{i=1, \dots, N} |v_i - w_i|,$$

where  $v$  and  $w$  are vectors of length  $N = 50$  to measure the distance between the values obtained by two methods. Our results are presented in Table 1.

**Table 1.** Difference measures.

$\alpha$	1.25		1.50		1.75	
$\beta$	$D_1 \cdot 10^{-9}$	$D_2 \cdot 10^{-9}$	$D_1 \cdot 10^{-9}$	$D_2 \cdot 10^{-9}$	$D_1 \cdot 10^{-9}$	$D_2 \cdot 10^{-9}$
-1.0	2.273	2.292	0.570	0.573	0.076	0.076
-0.5	2.252	2.262	0.569	0.570	0.076	0.076
0.0	2.239	2.239	0.567	0.567	0.076	0.076
0.5	2.234	2.243	0.566	0.567	0.076	0.076
1.0	2.236	2.255	0.566	0.568	0.076	0.076

We can see, that the method of two quadratures is more accurate than FFT-based methods with  $N = 2^{13}, 2^{16}$ . It gives (if  $\alpha \geq 1, 25$ ) at least nine accurate significant digits of the PDF values, while FFT-based methods show only five digits [10].

Direct numerical integration methods allow us to achieve greater precision, but at high cost of speed. For example, with adaptive direct numerical integration method (on AMD Athlon 2800+ 1.81 GHz processor) 750 density calculations with  $\varepsilon = 10^{-8}$  can be done in approximately 0.3 sec., however the accuracy of  $\varepsilon = 10^{-15}$  requires approximately 4.5 min. We compare the computational speed of the suggested method vs direct numerical integration methods: Nolan's method [12], adaptive integration method (3.1) and

Zolotarev integral representation method [11]. Zolotarev integral if  $\alpha \neq 1$ ,  $\zeta = -\beta \tan \frac{\pi\alpha}{2}$ ,  $\xi = -\frac{1}{\alpha} \arctan \zeta$  can be written as

$$p(x, \alpha, \beta) = \frac{\alpha(x - \zeta)^{\frac{1}{\alpha-1}}}{\pi|1 - \alpha|} \int_{-\xi}^{\frac{\pi}{2}} V(\theta, \alpha, \beta) e^{-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta, \alpha, \beta)} d\varphi, \quad x > \zeta,$$

where

$$V(\theta, \alpha, \beta) = (\cos \alpha\xi)^{\frac{1}{\alpha-1}} \left( \frac{\cos \theta}{\sin \alpha(\xi + \theta)} \right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\xi + (\alpha - 1)\theta)}{\cos \theta},$$

and

$$p(x, \alpha, \beta) = \begin{cases} \frac{\Gamma(\frac{1}{\alpha}) \cos \xi}{\pi \alpha (1 + \zeta^2)^{1/2\alpha}}, & x = \zeta, \\ p(-x, \alpha, -\beta), & x < \zeta. \end{cases}$$

Results of the comparison are presented in Table 2.

**Table 2.** Calculations of 750 PDF values from Table 1.

Method	Time, sec.
Nolan's, $\varepsilon = 10^{-9}$	1.2
Adaptive, $\varepsilon = 10^{-9}$	0.6
Zolotarev	0.6
Two quadratures	0.2

Thus, the method of two quadratures is faster than direct numerical integration methods and FFT-based methods in computational aspect and easily implementable. These features make the approach particularly useful in applications requiring a large number of fast and accurate PDF evaluations, such as maximum likelihood estimation of the parameters of financial series.

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