Mathematical Modelling and Analysis 2005. Pages 321–326
Proceedings of the 10th International Conference MMA2005&CMAM2, Trakai
 ISBN 9986-05-924-0

ON CONVEX OPTIMAL CONTROL PROBLEMS

V. AZHMYAKOV and W.H. SCHMIDT

Institute for Automation Technology, Department of Electrical Engineering, University of Magdeburg

PF 4120, D-39016 Magdeburg, Germany

Institute of Mathematics and Computer Sciences, University of Greifswald

Jahnstr. 15a, D-17487 Greifswald, Germany

E-mail: vadim.azhmyakov@e-technik.uni-magdeburg.de;

E-mail: wschmidt@uni-greifswald.de

Abstract. In this paper we propose a computational approach to a class of constrained optimal control problems (OCP's). We apply discrete approximation procedures and proximal-based regularization techniques to convex OCP's and describe how to carry out the numerical calculations in the context of convex programming. The presented approach makes it possible to obtain consistent approximate solutions.

Key words: optimal control, convex optimization, numerical methods

1. Introduction

We concentrate on the constrained OCP

$$\begin{aligned}
&\Phi(x(t_f)) = \min\ J(x(\cdot), u(\cdot))),\\
&\text{subject to } \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. on } [0, t_f],\ x(0) = x_0,\\
&u(t) \in U \text{ a.e. on } t \in [0, t_f],\\
&h(x(t_f)) \leq 0,\ q(t, x(t)) \leq 0\ \forall t \in [0, t_f],\ \int_0^{t_f} s(t, x(t), u(t))dt \leq 0.
\end{aligned} \tag{1.1}$$

Here, $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function,

$$f : [0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ h : \mathbb{R}^n \to \mathbb{R},$$
$$q : [0, t_f] \times \mathbb{R}^n \to \mathbb{R},\ s : [0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

and $x_0 \in \mathbb{R}^n$ is a fixed initial state. The initial OCP (1.1) contains target, state and integral constraints. Let us assume that the given functions h and

$q(t,\cdot)$, $t \in [0,t_f]$ are continuously differentiable. The function s is continuous and the control set U is a compact and convex subset of $\mathbb{R}^m$. Let

$$\mathcal{U} := \{u(\cdot) \in \mathbb{L}^2_m([0,t_f]) \ : \ u(t) \in U \text{ a.e. on } [0,t_f]\}$$

be the set of admissible control functions. Note that in the case of the convex control set U the set $\mathcal{U}$ is also convex. In addition, we assume that for each $u(\cdot) \in \mathcal{U}$ the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. on } [0,1], \\ x(0) = x_0 \end{cases} \tag{1.2}$$

has a unique absolutely continuous solution $x^u(\cdot)$. For some constructive existence/uniqueness conditions see e.g., [5, 12].

Let us introduce the mappings $\tilde{J}, \tilde{h}, \tilde{s} : \mathbb{L}^2_m([0,t_f]) \to \mathbb{R}$ and the mapping $\tilde{q} : \mathbb{L}^2_m([0,t_f]) \to \mathbb{C}([0,t_f])$ defined by

$$\begin{aligned} &\tilde{J}(u(\cdot)) := \Phi(x^u(t_f)), \quad \tilde{h}(u(\cdot)) := h(x^u(t_f)), \\ &\tilde{q}(u(\cdot))(t) := q(t, x^u(t)) \ \forall t \in [0,t_f], \\ &\tilde{s}(u(\cdot)) := \int_0^{t_f} s(t, x^u(t), u(t))\, dt. \end{aligned}$$

Evidently, the initial OCP (1.1) can be formulated as the following infinite-dimensional nonlinear program

$$\begin{aligned} &\text{minimize } \tilde{J}(u(\cdot)) \text{ subject to } u(\cdot) \in \mathcal{U}, \\ &\tilde{h}(u(\cdot)) \le 0, \ \tilde{q}(u(\cdot))(t) \le 0 \ \forall t \in [0,t_f], \ \tilde{s}(u(\cdot)) \le 0. \end{aligned} \tag{1.3}$$

In our paper we consider a class of the constrained OCP's (1.1) such that the corresponding problem (1.3) is a convex optimization problem in the Hilbert space $\mathbb{L}^2_m([0,t_f])$. The development of optimization theory has proceeded almost contemporarily with the systematical investigation of convex problems and their numerical treatment. A great amount of works is devoted to the theoretical and practical aspects of convex programming; see e.g., [7, 14] and the references therein. It is well known that the main classes of extremal problem include ill-posed problems [8]. Therefore, the use of standard optimization and discretization methods often proves to be unsuccessful for solving the ill-posed problems of the type (1.3). In the convex case we consider the techniques of the proximal-regularization (see e.g., [3, 8, 9, 10, 13]) and propose numerically stable computational schemes for the initial OCP(1.1). Note that in parallel with Tikhonov's regularization the proximal point algorithm is the main method for treating ill-posed problems of mathematical programming.

2. Convex Optimal Control Problems

Let us introduce the basic concept (see [4]).

Definition 1. We call the control system (1.2) a convex control system if for all $t \in [0, t_f]$ and all $k = 1, ..., n$ the functional $V_{k,t} : \mathcal{U} \to \mathbb{R}$

$$V_{k,t}(u(\cdot)) := x_k^u(t)$$

is convex.

We are interested in studying the convex control systems in the context of OCP's. Therefore, we give our next definition.

Definition 2. If the infinite-dimensional problem (1.3) is equivalent to a convex optimization problem in a real Hilbert space, then we call (1.1) a convex OCP.

We continue by considering the semilinear control systems, namely, the systems (1.2) with the right-hand side $f(t, x, u) = A(t)x + B(t, u)$.

Theorem 1. *Assume that* $A(t) = (a_{i,j}(t))_{j=1,\ldots,n}^{i=1,\ldots,n}$, $t \in [0, 1]$ *are regular* $n \times n$ *matrices and the functions* $a_{i,l}$ *are continuous. Let* $B : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ *be a continuous function and* $a_{i,j}(t) \geq 0$ *for all* $t \in [0, 1]$, $i, j = 1, ..., n$*. Suppose that functionals* $u(\cdot) \to B_k(t, u(t))$, $u(\cdot) \in \mathcal{U}$ *are convex for all indexes* $k = 1, ..., n$ *and all* $t \in [0, t_f]$*. Then the corresponding semilinear control system* (1.2) *is convex.*

Clearly, a linear control system (1.2) with

$$f(t, x, u) = \tilde{A}(t)x + \tilde{B}(t)u,$$

where $\tilde{A}(t) \in \mathbb{R}^{n \times n}$, $\tilde{B}(t) \in \mathbb{R}^{n \times m}$ are regular matrices, is also convex in the sense of Definition 1. Note that some important classical nonlinear differential equations, for example a controllable Bernoulli differential equation, can be reduced to a linear equation.

Definition 3. A functional $g : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ is called monotonically nondecreasing if $g(\xi) \geq g(\zeta)$ for all $\xi, \zeta \in \Gamma$ such that $\xi_k \geq \zeta_k$, $k = 1, ..., n$.

Clearly, the presented monotonicity concept can be expressed by introducing the positive cone $\mathbb{R}^n_{\geq 0}$ (the positive orthant). We now characterize a class of convex nonlinear control systems.

Theorem 2. *Assume that the function* f *in* (1.2) *is continuous and satisfies the following Lipschitz condition (uniformly in* $u \in U$*)*

$$||f(t, x_1, u) - f(t, x_2, u)|| \leq L||x_1 - x_2||, \quad \forall x_1, x_2 \in \mathbb{R}^n, \ u \in U,$$

where $t \in [0, t_f]$*. Let* $f_k(t, \omega)$, $k = 1, ..., n$ *be convex and monotonically nondecreasing functional with respect to the variable* $\omega := (x, u)$ *for every* $t \in [0, t_f]$*. Then the control system* (1.2) *is convex.*

Our next result establishes the convexity of an optimal control processes governed by a convex control system.

Theorem 3. *Let the control system in* (1.2) *be convex and the functionals* Φ, h, s *be convex and monotonically nondecreasing. Let the function* $q(t,\cdot)$ *be convex and monotonically nondecreasing for every* $t \in [0, t_f]$. *Then the associated* OCP (1.1) *is convex.*

The proofs of Theorem 1, Theorem 2 and Theorem 3 can be found in the work of authors [4].

3. The Constructive Computational Approach

We now deal with convex OCP's of the type (1.1). Let $N \in \mathbb{N}$ and

$$G_N := \{t_0 = 0, t_2, ..., t_N = t_f\}$$

be a (possible nonuniform) grid. We examine a finite-dimensional variant of problem (1.3)

$$\begin{aligned}&\text{minimize } \tilde{J}(u_N(\cdot)) \text{ subject to } u_N(\cdot) \in \mathcal{U} \cap \mathbb{L}_1^{2,N}(G_N),\\ &\tilde{h}(u_N(\cdot)) \leq 0,\ \tilde{q}(u_N(\cdot))(t_k) \leq 0\ \forall t_k \in G,\ \tilde{s}(u_N(\cdot)) \leq 0,\end{aligned} \tag{3.1}$$

where $k = 1, ..., N$. By $\mathbb{L}_1^{2,N}(G_N)$ we denote here the Euclidean space of piecewise constant control functions. Moreover, we consider a discretization of system (1.2). We use the Euler method for this purpose. It must be admitted that the first order Euler discretizations is particularly advantageous for relatively easy OCP's. Some alternative approximation procedures are described in [16].

Theorem 4. *Under the assumptions of* Theorem 3 *the discretized problem* (3.1) *is a convex minimization problem.*

The convex structure of problems (1.1) and (3.1) makes it possible to apply a proximal-based method and a standard optimization algorithm (see e.g., [11, 14]. The objective functional for the regularized problem (3.1) can be written in the form

$$\tilde{J}^r(u_N(\cdot)) := \tilde{J}(u_N(\cdot)) + \frac{\chi_l}{2}||u_N(\cdot) - u_N^l(\cdot)||^2_{\mathbb{L}_1^{2,N}(G_N)},\ l = 0, 1, ... ,$$

where $u_N^0(\cdot)$ is an admissible piecewise constant control and $\{\chi_l\}$ is a given sequence with $0 < \chi_l \leq C < \infty$. Note that the regularized objective functional $\tilde{J}^r$ is a strongly convex functional. Under some mild assumptions (see e.g., [8, 13]), the sequence $\{u_N(\cdot)\}$ generated by the classic proximal point algorithm is a minimizing sequence for (3.1). This sequence can be used for creating a strongly-convergent minimizing sequence [3]. The gradient $\nabla \tilde{J}(u^l(\cdot))$ of the unconstrained OCP (1.3) can be found as follows (see [1, 12, 15])

$$
\begin{aligned}
&\nabla \tilde{J}(u_N(\cdot))(t_k) = -H_u(t_k, x_N(t_k), u_N(t_k), p_N(t_{k+1})),\\
&x_N(t_{k+1}) = H_p(t_k, x_N(t_k), u_N(t_k), p_N(t_{k+1}),\ x_N(t_0) = x_0,\\
&p_N(t_k) = H_x(t_k, x_N(t_k), u_N(t_k), p_N(t_{k+1})),\ t_k \in G,\ k = 1, ..., N-1\ ,\\
&p_N(1) = -\Phi_x(x_N(t_N)),
\end{aligned}
\tag{3.2}
$$

where $x_N(\cdot)$ is the solution of a discretized state equation, $p_N(\cdot)$ are adjoint variables and

$$H(t_k, x_N(t_k), u_N(t_k), p_N(t_{k+1})) = \langle p_N(t_{k+1}), f(t_k, x_N(t_k), u_N(t_k)) \rangle_{\mathbb{R}^n}$$

is the Hamiltonian of (3.1). Moreover,

$$\nabla \tilde{J}^r(u_N(\cdot))(t_k) = \nabla \tilde{J}(u_N(\cdot))(t_k) + \chi_l(u_N(t_k) - u_N^l(t_k)),\ t_k \in G.$$

The discretized problem (3.1) can be solved directly by applying (3.2) and a gradient-based algorithm. We refer to [11, 14] for the general gradient-type methods, to [6] for the "gradient plus projection method" and to [12] for feasible directions algorithms.

4. Some Extensions

The investigation of the convex control systems involves a question of general interest. Let Ξ be a space of functions from $\mathbb{R}$ into $\mathbb{R}$ and Θ be a topological space. Let $\mathcal{T} : \Xi \times \Theta \to \mathbb{R}$ be a functional. Assume that for every $\theta \in \Theta$ the given equation $\mathcal{T}(\xi, \theta) = 0$ has a unique solution $\xi^\theta(\cdot) \in \Xi$. It is a familiar consideration in mathematics to seek to solve this equation for ξ, while viewing θ as a parameter. In the connection with the above theory of the convex control systems we can formulate the following problem: Conditions for the mapping $\mathcal{T}$ may be chosen whereby the functional $\mathcal{V} : \Theta \to \mathbb{R},\ \mathcal{V}(\theta) := \xi^\theta(t)$ is convex for every $t \in \mathbb{R}$.

Our results present a possible solution to the formulated general problem for $\Theta = \mathcal{U}$ and in the special case of the mapping $\mathcal{T}$ defined by the ordinary differential equations.

From the view-point of numerical mathematics we solve an OCP (1.1) approximately. Therefore, instead of the exact "equivalence" in the sense of Definition 2, one can consider a generalized concept of so-called approximately convex OCP's (see [1]). The application of this concept to β -relaxed OCP's (see [2]) is presented in [1].

In our paper we are concerned with the open-loop optimal control. The question of possible generalizations of our convexity results for the closed-loop solutions of OCP's is still an open question.

References

[1] V. Azhmyakov. *Numerically stable schemes for optimal control problems with constraints*. Habilitationsschrift, University of Greifswald, Greifswald, 2005.

[2] V. Azhmyakov and W.H. Schmidt. Approximations of relaxed optimal control problems. *Journal of Optimization Theory and Applications.* (to appear)

[3] V. Azhmyakov and W.H. Schmidt. Strong convergence of a proximal-based method for convex optimization. *Mathematical Methods of Operations Research*, **57**, 393 – 407, 2003.

[4] V. Azhmyakov and W.H. Schmidt. *Stable methods for convex optimal control problems with constraints.* Preprint-Reihe Mathematik University of Greifswald, Greifswald, 2005.

[5] L.D. Berkovitz. *Optimal control theory.* Springer, New York, 1974.

[6] A.A. Goldstein. Convex programming in Hilbert space. *Bull. Amer. Math. Soc.*, **70**, 709 – 710, 1964.

[7] J.B. Hiriart-Urruty and C. Lemarechal. *Convex analysis and minimization algorithms.* Springer, Berlin, 1993.

[8] A. Kaplan and R. Tichatschke. *Stable methods for ill-posed variational problems.* Akademie Verlag, Berlin, 1994.

[9] A. Kaplan and R. Tichatschke. Proximal point approach and approximation of variational inequalities. *SIAM Journal on Control and Optimization*, **39**, 1136 – 1159, 2000.

[10] B. Martinet. Regularisation d'inequations variationelles par approximations successives. *Revue Francaise Informat. Recherche Operationnelle*, **4**, 154 – 159, 1970.

[11] E. Polak. *Optimization.* Springer, New York, 1997.

[12] R. Pytlak. *Numerical methods for optimal control problems with state constraints.* Springer, Berlin, 1999.

[13] R.T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization*, **14**, 877 – 898, 1976.

[14] P. Spellucci. *Numerische verfahren der nichtlinearen optimierung.* Birkhauser, Basel, 1993.

[15] K.L. Teo, C.J. Goh and K.H. Wong. *A unifed computational approach to optimal control problems.* Wiley, New York, 1991.

[16] V.M. Veliov. Second order discrete approximations to linear differential inclusions. *SIAM Journal of Numerical Analysis*, **29**, 439 – 451, 1992.