Investigation of Spectrum for a Sturm–Liouville problem with Two-Point Nonlocal Boundary Conditions

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Important problems

- J.R. Cannon (1963),
  "The solution of the heat equation subject to specification of energy";

- L.I. Kamynin (1964),
  "A boundary value problem in the theory of the heat conduction with nonclassical boundary condition";

- A.V. Bitsadze and A.A. Samarskii (1969),
  "Some elementary generalizations of linear elliptic boundary value problems";

- N.I. Ionkin (1979),
  "A problem for the heat equation with a nonclassical (nonlocal) boundary condition";

- G. Infante (2003),
  "Eigenvalues of some non-local boundary-value problems."

- A. Ashyralyev (2008),
  "A note on the Bitsadze–Samariski type nonlocal boundary value problem in a Banach space."
R. Čiegis (1984), "The numerical solution of the heat equation with nonclassical condition";

R. Čiegis (1988), "Numerical solution of a problem with small parameter for the highest derivative and a nonlocal condition";


M.P. Sapagovas and A.D. Štikonas (2005), "On the structure of the spectrum of a differential operator with a nonlocal condition";

M.P. Sapagovas (2004), "Diferencialiniu lygčiu kraštiniai uždaviniai su nelokaliosiomis salygomis";

Important problems

Sturm–Liouville Problem with Nonlocal Boundary Conditions (NBCs):

- Are important for investigation of the existence and uniqueness of stationary problems solution [Ionkin 1996, Gulin 2003]
- Very complicated because are not self-adjoint
- Spectrum for such problems may be not positive (or real).
- Useful for investigation the stability of the finite difference schemes for nonstationary problems
- Useful for investigation of the convergence of iterative methods

S. Pečiulytė and A. Štikonas (2007), "Distribution of the critical and other points in boundary problems with nonlocal boundary condition";

S. Pečiulytė, O. Štikonienė and A. Štikonas (2008), "Investigation of Negative Critical Points of the Characteristic Function for Problems with Nonlocal Boundary Conditions";

A. Štikonas and O. Štikonienė (2009), "Characteristic Functions for Sturm–Liouville Problems with Nonlocal Boundary Conditions";


Aims and problems

In the case of the differential Sturm–Liouville Problem

\[-u'' = \lambda u, \quad t \in (0, 1),\]
\[u(0) = 0, \quad \text{or} \quad u'(0) = 0\]  \hspace{1cm} (1)

we investigate the following NBC:

\[u(1) = \gamma u(\xi),\]
\[u'(1) = \gamma u'(\xi),\]
\[u(\xi) = \gamma u(1 - \xi),\]

where \(\gamma \in \mathbb{R}\) ir \(\xi \in [0, 1]\).

Main problems:

- find Constant Eigenvalues, which do not depend on parameter \(\gamma\);
- find Zeroes, Poles and Critical Points of Characteristic Function;
- describe Spectrum Curves and investigate their properties;
- investigate the dependence of Spectrum Domain on parameter \(\xi\) in NBC, find bifurcation points and types.
Aims and problems

In the case of discrete Sturm–Liouville Problem we approximate differential equation by the following Finite-Difference Scheme (FDS)

\[ \frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = 1, n - 1. \] (3)

and investigate the following NBC:

\[ U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \quad U_n = \gamma U_m. \]

At the left side of interval one of the conditions was selected:

\[ U_0 = 0, \quad U_1 = U_0. \]

The discrete problem was obtained by approximating the differential problem by a finite difference scheme.

Main problems:

- find Constant Eigenvalues, which do not depend on parameter \( \gamma \);
- find Zeroes, Poles and Critical Points of Characteristic Function;
- determine the dependence of these points on the number of grid points;
- investigate the behavior of Spectral Curves in the neighborhood of special special points;
- find the quantitative relationships between the numbers of points mentioned.
SLP with one classical BC and another two-point NBC

\[-u'' = \lambda u, \quad t \in (0, 1),\]  
\[u(0) = 0,\]  
\[u(1) = \gamma u'(\xi),\]

where parameters \(\gamma \in \mathbb{R}\) and \(\xi \in [0, 1]\). The eigenvalue \(\lambda \in \mathbb{C}: = \mathbb{C}\) and eigenfunction \(u(t)\) can be complex function.

If \(\gamma = 0\), then we have the SLP with classical BCs. In this case eigenvalues and eigenfunction are known:

\[\lambda_k = (k\pi)^2, \quad u_k(t) = \sin(k\pi t), \quad k \in \mathbb{N}\]

The case \(\gamma = \infty\) corresponds to (4) with classical BCs \(u(0) = 0\) and \(u'(\xi) = 0\), \(\xi \in [0, 1]\), instead of condition (6) and eigenvalues and eigenfunction are:

\[\lambda_k = \left(\frac{(k - 1/2)\pi}{\xi}\right)^2, \quad u_k(t) = \sin\left(\frac{(k - 1/2)\pi t}{\xi}\right), \quad k \in \mathbb{N}.\]
Figure 1: Bijective mapping $\lambda = (\pi q)^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$.

Nontrivial solution of the problem (4)–(6) exists if $q \in \mathbb{C}_q$ is the root of a equation

$$\frac{\sin(\pi q)}{\pi q} = \gamma \cos(\xi \pi q) \quad Z(q) = \gamma P_\xi(q). \quad (9)$$

We will define a Constant Eigenvalue (CE) as the eigenvalue which does not depend on the parameter $\gamma$. Then for any CE $\lambda \in \mathbb{C}_\lambda$ there exists the Constant Eigenvalue Point (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \quad P_\xi(q) = 0, \quad (10)$$
\( \lambda = 0 \)

Eigenvalue \( \lambda = 0 \) exists if and only if \( \gamma = 1 \).

Lemma 1.6.

For SLP (4)—(6) Constant Eigenvalues exist only for rational parameter \( \xi = m/n \in (0, 1) \), \( m \in \mathbb{N}_o \), \( n \in \mathbb{N}_e \), values and those eigenvalues are equal to \( \lambda_s = (\pi c_s)^2 \), \( c_s := (s - 1/2)n \), \( s \in \mathbb{N} \).

For SLP (4)—(6) we have meromorphic Complex Characteristic Functions (Complex CF):

\[
\gamma_c(q) = \gamma_c(q; \xi) := \frac{Z(q)}{P_{\xi}(q)} = \frac{\sin(\pi q)}{\cos(\xi \pi q)}, \quad z \in \mathbb{C},
\]

(11)

\( \gamma \)-points of Complex CF define EPs (and Eigenvalues, too) which depend on parameter \( \gamma \). We call such EPs Nonconstant Eigenvalue’s Points (NEP) and corresponding Eigenvalues as Nonconstant Eigenvalues.
Remark 1.1.

If the parameter $\xi = 0$, then from the formula (9) we obtain that $P_\xi \equiv 1$. So, $\mathcal{Z}_\xi = \emptyset$ and CEPs do not exist. If $\xi = 1$ then there are no CEPs, because the functions $\sin(\pi q)$ and $\cos(\pi q)$ have no common zeroes (we have the third type BC).

Remark 1.2.

If the parameter $\xi \notin \mathbb{Q}$, then CEPs do not exist, because the equation $\xi l = k - \frac{1}{2}$ has not roots for $l, k \in \mathbb{N}$.

Remark 1.3.

If $\xi \in \mathbb{Q}$, $\xi = m/n$ and $n \in \mathbb{N}_e$ then the right hand side of equation $nk - lm = \frac{n}{2}$ is integer number. If $n \in \mathbb{N}_o$ then this equation has no roots.

Remark 1.7.

In the case $\xi = 0$ function $P_\xi \equiv 1$ and PPs do not exist. If $\xi > 0$ a set of poles $\mathcal{P}_\xi = \emptyset$ or countable. So, PPs exist if $\xi \neq 1/n$. 
(a) CF ($\xi = 0$)  (b) Spectrum Curves ($\xi = 0$)  (c) Real CF ($\xi = 0$)

(d) CF ($\xi = 1$)  (e) Spectrum Curves ($\xi = 1$)  (f) Real CF ($\xi = 1$)

Figure 2: CCF, Spectrum Domain, Real CF for $\xi = 0, \xi = 1$. ● – Zero Point, ○ – Pole Point, ◦ – Ramification Point, ∗ – Branch Eigenvalue Point, ● – Critical Point.
Real Characteristic Function (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$. We can use the argument $x \in \mathbb{R}$ for Real CF:

$$
\gamma_r(x) = \gamma_r(x; \xi) := \begin{cases} 
\gamma(-ix; \xi) = \frac{\sinh(\pi x)}{\pi x \cosh(\xi \pi x)}, & x \leq 0; \\
\gamma(x; \xi) = \frac{\sin(\pi x)}{\pi x \cos(\xi \pi x)}, & x \geq 0.
\end{cases} \quad (12)
$$

This function is useful for investigation of real negative, zero and positive eigenvalues

$$
\lambda = \lambda_r(x) = \lambda_r(x; \xi) := \begin{cases} 
-(\pi x)^2, & x \leq 0; \\
(\pi x)^2, & x \geq 0.
\end{cases} \quad (13)
$$

Remark 1.11.

In the case $\xi = \xi_c = \frac{1}{\sqrt{3}}$ the point $q = 0$ is 3CP in the domain $\mathbb{C}_q$, but for $\lambda = 0$ it is only 1CP, because $q = 0$ is RP for map $\lambda = (\pi q)^2$. In the complex plane $\mathbb{C}_\lambda$ the Taylor series CF $\gamma(q)$ have a form

$$
\gamma(\lambda, \xi) := 1 + (-\frac{1}{6} + \frac{1}{2} \xi^2)\lambda + (\frac{1}{120} - \frac{1}{24} \xi^4 - (\frac{1}{2} (\frac{1}{6} - \frac{1}{2} \xi^2))\xi^2)\lambda^2 + O(\lambda^3). \quad (14)
$$

If $\xi \neq \xi_c$, then point $q = 0$ and $\lambda = 0$ are not CPs.
Lemma 1.12.

Zero Point of CF can not be CP.

Remark 1.13.

Pole Point of CF is not CP. Function $\gamma^{-1}$ has CP at this point only if order of the pole is greater than the first.

Remark 1.14.

In the case $\xi = 1$ and $\gamma \neq 0$ we can consider boundary condition $u'(1) = \tilde{\gamma}u(1)$, $\tilde{\gamma} \in \mathbb{R}$, where $\tilde{\gamma} = \gamma^{-1}$. Now CF is $\tilde{\gamma} = \pi q \cos(\pi q)/\sin(\pi q)$ and its zeroes are $\tilde{z}_k = p_k$, $k \in \mathbb{N}$, poles are $\tilde{p}_k = z_k$, $k \in \mathbb{N}$ (for $\xi = 1$ CEPs do not exist, but in the general case $\tilde{c}_k = c_k$ for all $k$). For parameter $\tilde{\gamma} \in \mathbb{R}$ all Spectrum Curves will be regular.

Remark 1.9.

A point $q = \infty \notin \mathbb{C}_q$. This point is singular (isolated essential point if $\mathcal{P}_\xi = \emptyset$, otherwise we have cluster of poles) point.
Zero and Pole bifurcation type $\beta_{ZP}$

Figure 3: Spectrum Curves for various parameter $\xi$ values.

- Critical Point at Branch Eigenvalue Point.
The second order CP bifurcation $\beta_{2B}$

Figure 4: Spectrum Curves for various parameter $\xi$ values, bifurcations.

(a) $\xi = 0.247$
(b) $\xi = \frac{1}{4}$
(c) $\xi = 0.25026$
(d) $\xi = \xi_{2b} \approx 0.25028$
(e) $\xi = 0.2503$
(f) $\xi = 0.253$
The second order CP bifurcation $\beta_{2B}$

Figure 5: Spectrum Curves for various parameter $\xi$ values, bifurcations.
Conclusions of this Chapter

1. For SLP (4)–(6) CEs do not exist for irrational parameter $\xi$ and exist only for $\xi = \frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$.

2. SLP (4)–(6) has two types CPs: the first, the second order. We have only one 3CP, $b_{2,1} = 0$, $\xi = \xi_c = 1/\sqrt{3}$. But this point is 1CP in the domain $\mathbb{C}_\lambda$. The negative CP exists only for $\xi > \xi_c$.

3. For SLP (4)–(6) we obtain two types’ bifurcations:
   - $\beta_{ZP} : (z_{l,s}, p_{k_s}) \rightarrow c_s \rightarrow (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ when zero and pole of CF merge into CEP and we get a loop type curve.
   - $\beta_{2B} : (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \rightarrow b_{l_s-1,l_s+1,l_s} \rightarrow \emptyset$ when two 1CPs merge into one 2CP. At this bifurcation the loop type curve vanish.
Let us analyze SLP with one classical BC

\[-u'' = \lambda u, \quad t \in (0, 1),\]

\[u(0) = 0,\]  \hspace{1cm} (15) \hspace{1cm} (16)

and another two-point NBC of Samarskii–Bitsadze type:

\[u'(1) = \gamma u(\xi), \quad \text{Case 1} \]

\[u'(1) = \gamma u'(\xi), \quad \text{Case 2} \]  \hspace{1cm} (17a) \hspace{1cm} (17b)

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$.


If $\gamma = 0$, we have problems with classical BCs. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter $\xi$:

\[\lambda_k = \pi^2 (k - 1/2)^2, \quad u_k = \sin(\pi(k - 1/2)t), \quad k \in \mathbb{N}.\]  \hspace{1cm} (18)

Kristina Bingelė, Sigita Pečiulytė, Artūras Štikonas (2009); “Investigation of complex eigenvalues for stationary problems with two-point nonlocal boundary condition”.
There exists a nontrivial solution (eigenfunction) if $q$ is the root of the function:

$$
\cos(\pi q) = \gamma \frac{\sin(\xi \pi q)}{\pi q}, \quad Z(q) = \gamma P_\xi(q) \quad q \in \mathbb{C}, \quad (19a)
$$

$$
\cos(\pi q) = \gamma \cos(\xi \pi q), \quad Z(q) = \gamma P_\xi(q) \quad q \in \mathbb{C}. \quad (19b)
$$

**Lemma 2.3.**

For SLP (15)–(17a) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0, 1)$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := (s - 1/2)n$, $s \in \mathbb{N}$.

**Lemma 2.4.**

For SLP (15)–(17b) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0, 1)$, $m, n \in \mathbb{N}_o$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.
For SLP (4)–(6) we have meromorphic Complex Characteristic Functions (Complex CF)

\[
\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\pi q \cos(\pi q)}{\sin(\xi \pi q)}, \quad q \in \mathbb{C}_q, \quad \text{(20a)}
\]

\[
\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\cos(\pi q)}{\cos(\xi \pi q)}, \quad q \in \mathbb{C}_q. \quad \text{(20b)}
\]

**Real Characteristic Function** (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF \( \gamma_c(q) \) on the set \( \mathbb{R}_q \):

\[
\gamma_r(x) = \gamma_r(x; \xi) = \begin{cases} 
\frac{\pi x \cosh(\pi x)}{\sinh(\xi \pi x)}, & x \leq 0; \\
\frac{\pi x \cos(\pi x)}{\sin(\xi \pi x)}, & x \geq 0.
\end{cases} \quad \text{(21a)}
\]

\[
\gamma_r(x) = \gamma_r(x; \xi) = \begin{cases} 
\frac{\cosh(\pi x)}{\cosh(\xi \pi x)}, & x \leq 0; \\
\frac{\cos(\pi x)}{\cos(\xi \pi x)}, & x \geq 0.
\end{cases} \quad \text{(21b)}
\]
Problem with Dirichlet condition
Problem with Neumann condition
Problem with one symmetrical type NC

(a) Real CF
(b) Spectrum Curves
(c) $\xi = 0$ (the limit case)

Figure 6: Real CF and Spectrum Curves for $\xi = 1$ and Real CF for $\xi = 0$ in Case 1.

(a) Real CF $\xi = 0$
(b) Real CF $\xi = \frac{499}{500}$
(c) Real CF $\xi = 1$ (the limit case)

Figure 7: Real CF for $\xi = 0$ and $\xi \lesssim 1$ in Case 2.
Bifurcations $\beta_{ZP}^{-1}$ and $\beta_{2B}^{-1}$ in Case 1.

Figure 8: Spectrum Curves for various parameter $\xi$ values in Case 1.
Symmetric Zero and Pole bifurcation $\beta_{ZP}^0$ in Case 2.

Figure 9: CF and bifurcation in Case 2.
Let us investigate SLP

\[-u'' = \lambda u, \quad t \in (0, 1), \quad (22)\]

with one classical (Neumann type) BC:

\[u'(0) = 0, \quad (23)\]

and another two-point NBC \((0 \leq \xi \leq 1)\):

\[u'(1) = \gamma u(\xi), \quad (24a)\]
\[u'(1) = \gamma u'(\xi), \quad (24b)\]
\[u(1) = \gamma u'(\xi), \quad (24c)\]
\[u(1) = \gamma u(\xi), \quad (24d)\]

where parameters \(\gamma \in \mathbb{R}\) and \(\xi \in [0, 1]\).

If \(\gamma = 0\), we obtain classical BVP. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter \(\xi\):

\[\lambda_k = (\pi k)^2, \quad u_k = \cos(\pi k t), \quad k \in \mathbb{N} \quad (\text{Case 1 and 2}), \quad (25a)\]
\[\lambda_k = \pi^2 (k - 1/2)^2, \quad u_k = \cos(\pi (k - 1/2) t), \quad k \in \mathbb{N} \quad (\text{Case 3 and 4}). \quad (25b)\]

**Theorem 2.9.**

Spectra for SLPs \((15)-(17b)\) and \((22)-(24d)\) overlap for all \(\gamma\) and \(\xi\).
Eigenvalue $\lambda = 0$ exists if and only if:

- $\gamma = 0$ in Case 1;
- $\gamma$ is any number in Case 2;
- $\gamma = 1$ in Case 4.

In Case 3 eigenvalue $\lambda = 0$ does not exist.

There exists a nontrivial solution (eigenfunction) if $q$ is the root of the function:

\begin{align}
-\pi q \sin(\pi q) &= \gamma \cos(\pi q \xi), \\
q \sin(\pi q) &= \gamma q \sin(\pi q \xi), \\
-\cos(\pi q) &= \gamma \pi q \sin(\pi q \xi), \\
\cos(\pi q) &= \gamma \cos(\pi q \xi). 
\end{align}

We see that (26d) in Case 4 is the same as (19b) in Case 2.

Kristina Skučaitė-Bingelė and Artūras Štikonas (2011); "Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition".

Artūras Štikonas and Olga Štikonienė (2009); "Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions" (Case 2 (26b)).
We introduce two entire functions:

\[
Z(q) := \pi q \sin(\pi q); \quad P_\xi(q) := -\cos(\xi \pi q), \quad q \in \mathbb{C}, \quad (27a)
\]
\[
Z(q) := \pi q \sin(\pi q); \quad P_\xi(q) := \pi q \sin(\pi q \xi), \quad q \in \mathbb{C}, \quad (27b)
\]
\[
Z(q) := \cos(\pi q); \quad P_\xi(q) := -\pi q \sin(\pi q \xi), \quad q \in \mathbb{C}. \quad (27c)
\]

For any CE \( \lambda \in \mathbb{C}_\lambda \) there exists the \textit{Constant Eigenvalue Point} (CEP) \( q \in \mathbb{C}_q \). CEP are roots of the system:

\[
Z(q) = 0, \quad P_\xi(q) = 0. \quad (28)
\]
Corollary 2.10.

Spectrum Curves and Spectrum Domain $\mathcal{N}_\xi$ for SLPs (15)–(17b) and (22)–(24d) are the same.

Remark 2.13.

In Case 2 RP $q = 0$ is CEP (of the second order). For the other cases CEPs are positive.

Remark 2.18.

CEP at Ramification Point $c_0 = 0$ in Case 2 is double in $\mathbb{C}_q$ but corresponding CE $\lambda = 0$ is simple.
Lemma 2.15.

For SLP (22)–(24a) Constant Eigenvalues exist only for rational parameter \( \xi = m/n \in (0, 1) \), \( m \in \mathbb{N}_o \), \( n \in \mathbb{N}_e \), \( \gcd(m, n) = 1 \), values and those eigenvalues are equal to \( \lambda_s = (\pi c_s)^2 \), \( c_s := n(s - 1/2) \), \( s \in \mathbb{N} \).

Lemma 2.16.

For SLP (22)–(24b) Constant Eigenvalues exist only for rational parameter \( \xi = m/n \in (0, 1) \), \( m, n \in \mathbb{N} \), \( \gcd(m, n) = 1 \), values and those eigenvalues are equal to \( \lambda_s = (\pi c_s)^2 \), \( c_s := ns \), \( s \in \mathbb{N}_0 \).

Lemma 2.17.

For SLP (22)–(24c) Constant Eigenvalues exist only for rational parameter \( \xi = m/n \in (0, 1) \), \( m \in \mathbb{N}_e \), \( n \in \mathbb{N}_o \), \( \gcd(m, n) = 1 \), values and those eigenvalues are equal to \( \lambda_s = (\pi c_s)^2 \), \( c_s := n(s - 1/2) \), \( s \in \mathbb{N} \).
For SLP (22)–(24) we have meromorphic Complex Characteristic Functions (Complex CF)

\[ \gamma_c(q) = -\frac{\pi q \sin(\pi q)}{\cos(\xi \pi q)}, \]  
(29a)

\[ \gamma_c(q) := \frac{\sin(\pi q)}{\sin(\xi \pi q)}, \]  
(29b)

\[ \gamma_c(q) := -\frac{\cos(\pi q)}{\pi q \sin(\xi \pi q)}. \]  
(29c)

**Theorem 2.25.**

Spectrum of SLP (22)–(24) has one additional simple eigenvalue \( \lambda = 0 \) in addition to eigenvalues of spectrum of SLP (22)–(24) in Introduction [A. Štikonas and O. Štikonienė 2009] for all \( \gamma \) and \( \xi \in (0, 1) \).

**Corollary 2.26.**

Additional eigenvalue \( \lambda = 0 \) for SLP (22)–(24b) corresponds to nonregular Spectrum Curve (CEP \( q = 0 \)) \( \mathcal{N}_0 \). The other Spectrum curves for both SLPs overlap.
Bifurcations $\beta_{ZP}^{-1}$ and $\beta_{2B}^{-1}$ in Case 1.

Figure 10: Bifurcations in Case 1 (Neumann BC).

(a) $\xi = 0.4885$

(b) $\xi = 0.4998$

(c) $\xi = \xi_{2b} = 0.4998645…$

(d) $\xi = 0.499867$

(e) $\xi = \xi_c = 1/2$

(f) $\xi = 0.5001$
Symmetric Zero and Pole bifurcation $\beta^0_{ZP}$ in Case 2.

Figure 11: CF and bifurcation in Case 2 (Neumann BC).

(d) $\xi = 0.499999999$

(e) $\xi = \xi_c = 1/2$

(f) $\xi = 0.500000001$
Bifurcations $\beta_{ZP}$ and $\beta_{2B}$ in Case 3.

(a) $\xi = 0.3999$

(b) $\xi = \xi_c = 2/5$

(c) $\xi = 0.4001$

(d) $\xi = \xi_{2b} = 0.400218...$

(e) $\xi = 0.400235$

(f) $\xi = 0.40036$

Figure 12: Bifurcations in Case 3 (Neumann BC).
(a) $\xi = 0.6656$

(b) $\xi = \xi_c = \frac{2}{3}$

(c) $\xi = 0.6676$

(d) $\xi = 0.673$

(e) $\xi = \xi_{2b} = 0.67495…$

(f) $\xi = 0.678$

Figure 13: Bifurcations in Case 3 (Neumann BC).
Let us analyze the SLP with one classical BC

\[-u'' = \lambda u, \quad t \in (0, 1),\]
\[u(0) = 0,\]

and another two-point NC

\[u(\xi) = \gamma u(1 - \xi),\]

with the parameters \(\gamma \in \mathbb{R}\) and \(\xi \in [0, 1]\).

**Case \(\gamma = 0\).** If \(\xi = 0\), then have problem (30),(31) with one BC \(u(0) = 0\) only. If \(0 < \xi \leq 1\), then we have the classical BVP in the interval \([0, \xi]\) with BCs \(u(0) = 0, u(\xi) = 0\), and its eigenvalues and eigenfunctions are

\[\lambda_k = \left(\frac{\pi k}{\xi}\right)^2, \quad u_k(t) = \sin \left(\frac{\pi k t}{\xi}\right), \quad k \in \mathbb{N}.\]

**Case \(\gamma = \infty\).** If \(\xi = 1\), we have problem (30)–(31) with one BC \(u(0) = 0\). If \(0 \leq \xi < 1\) then we have the same situation as in Case \(\gamma = 0\) with the BVP in the interval \([0, 1 - \xi]\).

**Case \(\xi = \frac{1}{2}\).** If \(\gamma = 1\), then we have problem (30)–(31) with one BC \(u(0) = 0\). If \(\gamma \neq 1\), then we have BVP in the interval \([0, \frac{1}{2}]\) and the initial value problem in the interval \([\frac{1}{2}, 1]\).
There exists a nontrivial solution, if $q$ is the root of the equation

$$\frac{\sin(\xi \pi q)}{\pi q} = \gamma \frac{\sin ((1 - \xi) \pi q)}{\pi q}. \quad Z_\xi(q) = \gamma P_\xi(q) = \gamma Z_{1-\xi}(q), \quad q \in \mathbb{C}_q. \quad (34)$$

Roots of the system

$$\sin(\pi q) = 0, \quad \sin(\pi q \xi) = 0, \quad (35)$$

are CEP of SLP (30)–(32).

Lemma 2.29.

For SLP (30)–(32) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0, 1)$, $m, n \in \mathbb{N}$, $\xi \neq 1/2$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s = ns$, $s \in \mathbb{N}$.

Kristina Skučaitė-Bingelė, Artūras Štikonas (2013); "Inverstigation of the spectrum for Sturm-Liouville problems with a nonlocal boundary condition".
For SLP (30)–(32) we have meromorphic Complex CF

$$\gamma_c(q) = \frac{Z(q)}{Z_{1-\xi}(q)} = \frac{\sin(\xi \pi q)}{\sin ((1 - \xi) \pi q)}, \quad q \in \mathbb{C}_q.$$  \hspace{1cm} (36)

**Remark 2.31.**

NC (32) we can rewrite as

$$u(1 - \xi) = \tilde{\gamma} u(\xi), \quad \tilde{\gamma} = 1/\gamma.$$ \hspace{1cm} (37)

NC (37) we can rewrite as

$$u(\tilde{\xi}) = \tilde{\gamma} u(1 - \tilde{\xi}).$$ \hspace{1cm} (38)

So, the spectrum for SPL (30)–(32) with parameters $0 < \xi < 1/2$ and $\gamma$ is the same as the spectrum for SPL (30)–(31), (37) with parameters $1/2 < \tilde{\xi} < 1$ and $\tilde{\gamma} = 1/\gamma$. Thus, it is enough to investigate problem (30)–(32) with the parameter $1/2 < \xi < 1$.  

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Problem with one symmetrical type NC

Theorem 2.33.

Spectrum for SLP (30)–(32) for $1/2 < \xi < 1$ is equivalent to Spectrum for SLP:

\[-u'' = \lambda u, \quad t \in (0, 1),\]

\[u(0) = 0,\]

\[u(1) = \gamma u(\xi),\]  

Artūras Štikonas (2007);
"The Sturm–Liouville problems with nonlocal boundary conditions"
Conclusions of this Chapter

1. For SLP with Dirichlet type BC (two cases $\text{SLP}_1^d$, $\text{SLP}_2^d$) (15)–(17), SLP with Neumann type BC (three cases $\text{SLP}_1^n$, $\text{SLP}_2^n$, $\text{SLP}_3^n$ and $\text{SLP}_4^n \sim \text{SLP}_2^d$) (22)–(24) and SLP with symmetrical type BC ($\text{SLP}^s$) (30)–(32) CEs do not exist for irrational parameter $\xi$ and exist only for rational $\xi = \frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $\gcd(m, n) = 1$

2. CPs of the first order (and complex eigenvalues) exist for all SLP in case $\xi \in (0, 1)$, for $\text{SLP}_2^d$ in case $\xi = 0$, $\text{SLP}_1^n$ in case $\xi = 0$. We have infinite number of such CPs of the first order. Negative CP of the first order exists for $\text{SLP}_1^n$ in case $\xi \in (0, 1)$. CPs of the second order exist for $\text{SLP}_2^d$, $\text{SLP}_1^n$, $\text{SLP}_3^n$ (only for some $\xi \in (0, 1)$).

3. For SSLP$_1^d$, SSLP$_2^d$, SSLP$_1^n$, SSLP$_2^n$, SSLP$_3^n$, SLPS we obtain five types bifurcations ($\beta^0_{ZP}$, $\beta_{2B}$, $\beta_{ZP}$, $\beta_{2B}^{-1}$ and $\beta_{ZP}^{-1}$)
In this chapter we investigate a *discrete Sturm–Liouville Problems* (dSLP) corresponding to SLPs in Chapter 1 and Chapter 2:

\[-u'' = \lambda u, \quad t \in (0, 1),\]  

(40)

with one classical (Dirichlet or Neumann) BC:

\[u(0) = 0 \text{ or } u'(0) = 0\]  

(41)

and another two-point NBC:

\[u(1) = \gamma u' (\xi),\]  

(42a)

\[u(1) = \gamma u (\xi),\]  

(42b)

with the parameter \(\gamma \in \mathbb{R}, 0 < \xi < 1\).

Kristina Skučaitė-Bingelė, Artūras Štikonas (2011); "Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition”.

Kristina Bingelė, Agnė Bankauskienė, Artūras Štikonas (2019); "Spectrum Curves for a discrete Sturm–Liouville problem with one integral boundary condition".
We approximate SLP (40)–(42) with Dirichlet BC by the following Finite-Difference Scheme (FDS) and get a discrete Sturm–Liouville Problem (dSLP):

\[
\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = 1, n - 1, \quad (43)
\]

\[
U_0 = 0, \quad (44)
\]

with NBC (0 < m < n)

\[
U_n = \frac{\gamma}{2h} (U_{m+1} - U_{m-1}), \quad (45a)
\]

\[
U_n = \gamma U_m. \quad (45b)
\]

If \(\gamma = 0\), we have the classical BCs and all the \(n - 1\) eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters \(\xi\):

\[
\lambda_k(0) = \lambda^h(q_k(0)), \quad U_{k,j}(0) = \sin(\pi q_k(0)t_j), \quad q_k(0) = k \in \mathbb{N}^h. \quad (46)
\]
\[
\frac{\sin(\pi q)}{\pi q} \cdot \frac{\pi q h}{\sin(\pi q h)} = \gamma \cos(\xi \pi q), \quad (47a)
\]
\[
\frac{\sin(\pi q)}{\pi q} \cdot \frac{1}{1 - h q} = \gamma \frac{\sin(\xi \pi q)}{\pi q} \cdot \frac{1}{1 - h q}. \quad (47b)
\]

Roots of this equation are EPs for dSLP (43)–(45). The bijection
\[
\lambda = \lambda^h(q) = \frac{4}{h^2} \sin^2(\pi q h/2) = \frac{2}{h^2} (1 - \cos(\pi z h))
\]
allows to find corresponding eigenvalues.

We introduce functions:
\[
Z^h(z) := Z(z) \cdot \frac{\pi z h}{\sin(\pi z h)}, \quad Z(z) := \frac{\sin(\pi z)}{\pi z}, \quad P^h_\xi(z) = P_\xi(z) := \cos(\xi \pi z) \quad (48a)
\]
\[
Z^h(z) := Z(z) \cdot \frac{1}{\pi z(h z - 1)}, \quad Z(z) := \sin(\pi z), \quad P^h_\xi(z) = P_\xi(z) \cdot \frac{1}{\pi z(h z - 1)}, \quad (48b)
\]
\[
P_\xi(z) := \sin(\xi \pi z);
\]

For any CE \( \lambda \in \mathbb{C}_\lambda \) there exists the \textbf{Constant Eigenvalue Point (CEP) } \( q \in \mathbb{C}_q \). CEP are roots of the system:
\[
Z^h(q) = 0, \quad P^h_\xi(q) = 0. \quad (49)
\]
Lemma 3.5.

For dSLP (43)–(45a) Constant Eigenvalues exist only for $\xi = M/N \in (0, 1)$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := (s - 1/2)N$, $s = 1, K$.

Lemma 3.6.

For dSLP (43)–(45b) Constant Eigenvalues exist only for $\xi = M/N \in (0, 1)$, $M, N \in \mathbb{N}$, $K > 1$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := Ns$, $s = 1, K - 1$.

For dSLP (43)–(45) we have meromorphic functions (Complex CF)

\[
\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} \cdot \frac{\pi q h}{\sin(\pi q h)} = \frac{\sin(\pi q)}{\pi q \cos(\xi \pi q)} \cdot \frac{\pi q h}{\sin(\pi q h)}, \quad (50a)
\]

\[
\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\sin(\xi \pi q)} \cdot \hspace{1cm} (50b)
\]
Real CF in Case 1.

Figure 14: Real CF (dSLP Dirichlet BC) for various parameter ξ values in Case 1.
Figure 15: Spectrum Curves (dSLP Dirichlet BC) for various $\xi$ values in Case 1.
Discrete Problem with Dirichlet condition

(a) $\xi = 36/63$

(b) $\xi = 4/7$

(c) $\xi = 37/63$

(d) $\xi = 300/400$

(e) $\xi = 301/400$

(f) $\xi = 302/400$

Figure 16: Spectrum Curves near Ramification Point $q = 0$ Case 1.
Real CF in Case 2.

Figure 17: Real CF (dSLP Dirichlet BC) for various parameter $\xi$ values in Case 2.
Figure 18: Spectrum Curves (dSLP Dirichlet BC) for various $\xi$ values in Case 2.
We approximate SLP (40)–(42) with Neumann type BC by FDS and get dSLP:

\[
\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = 1, n - 1, \\
U_0 = U_1,
\]

(51)

with NBC \((0 < m < n)\)

\[
U_n = \frac{\gamma}{2h} (U_{m+1} - U_{m-1}), \\
U_n = \gamma U_m.
\]

(53a, 53b)

If \(\gamma = 0\), we have the classical BCs and all the \(n - 1\) eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters \(\xi\):

\[
\lambda_k(0) = \frac{4}{h^2} \sin^2(\pi q_k(0)h/2), \quad U_{k,j}(0) = \frac{\cos(\pi q_k(0)(t_j - h/2))}{\cos(\pi q_k(0)h/2)}, \\
k = 1, n - 1.
\]

(54)
\[-\frac{\cos(\pi q(1 - h/2))}{\cos(\pi qh/2)} = \gamma \frac{\sin(\pi q(\xi - h/2)) \cdot \sin(\pi qh/2)}{h/2},\]  
\[\frac{\cos(\pi q(1 - h/2))}{\cos(\pi qh/2)} = \gamma \frac{\cos(\pi q(\xi - h/2))}{\cos(\pi qh/2)}.\]  

Roots of this equation are EPs for dSLP (51)–(53).

We introduce functions:

\[Z^h(z) := \frac{\cos(\pi z(1 - h/2))}{\cos(\pi zh/2)};\]  
\[P^h_\xi(z) := -\sin(\pi z(\xi - h/2)) \cdot \frac{\sin(\pi zh/2)}{h/2},\]  
\[P^h_\xi(z) := \frac{\cos(\pi z(\xi - h/2))}{\cos(\pi zh/2)}.\]

For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the Constant Eigenvalue Point (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

\[Z^h(q) = 0, \quad P^h_\xi(q) = 0.\]
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Discrete Problem with Dirichlet condition
Discrete Problem with Neumann condition

Lemma 3.17.
For dSLP (51)–(53a) Constant Eigenvalues do not exist.

Lemma 3.18.
For dSLP (51)–(53b) Constant Eigenvalues exist only for \( \xi = m/n \in (0, 1) \), \( K > 1 \), values and those eigenvalues are equal to \( \lambda_s = \lambda^h(c_s) \),
\[
c_s := n/K \cdot (2s - 1), \quad s = 1, (K - 1)/2.
\]

For dSLP (51)–(53) we have meromorphic functions (Complex CF)
\[
\gamma_c(q) := -\frac{\cos \left( \pi q(1 - h/2) \right)}{\sin \left( \pi q(\xi - h/2) \right)} \cdot \frac{h}{\sin(\pi qh)}, \quad (59a)
\]
\[
\gamma_c(q) := \frac{\cos \left( \pi q(1 - h/2) \right)}{\cos \left( \pi q(\xi - h/2) \right)} \cdot \quad (59b)
\]
Real CF in Case 1.

Figure 19: Real CF $\gamma_r$ (dSLP Neumann BC) for various parameter $\xi$ values in Case 1.
Figure 20: Spectrum Curves (dSLP Neumann BC) for various $\xi$ values in Case 1.
Real CF in Case 2.

Figure 21: Real CF $\gamma_r$ (dSLP Neumann BC) for various parameter $\xi$ values in Case 2.
Figure 22: Spectrum Curves (dSLP Neumann BC) for various $\xi$ values in Case 2.
We investigate the spectrum discrete Problems: two dSLP with one classical Dirichlet BC and two-points NBCs, two dSLP with one classical Neumann type BC and two-points NBCs.

1. For dSLP with one classical Dirichlet BC CEs can exist (in Case 1 and Case 2);
2. For dSLP with one classical Neumann BC CEs can exist (in Case 1) and do not exist (in Case 2);
3. In limit case, the discrete Problems CF is the same as for differential SLPs.