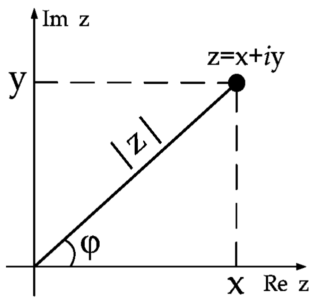
2 užsiėmimas. Kompleksiniai skaičiai



Kompleksiniu skaičiumi vadina
$z = x + iy$, kur $i^2 = -1$.
Realioji skaičiaus z dalis Re $z = x$.
Menamoji skaičiaus z dalis Im $z = y$.
Skaičiaus z modulis $|z| = \sqrt{x^2 + y^2}$,
argumentas $\arg z = \varphi$,
Arg $z = \arg z + 2\pi k, \quad k \in \mathbb{Z}$

$$\arg z = \begin{cases} \text{arctg}\,\frac{y}{x}, & x > 0; \\ \text{arctg}\,\frac{y}{x} + \pi, & x < 0, \quad y \geqslant 0; \\ \text{arctg}\,\frac{y}{x} - \pi, & x < 0, \quad y < 0. \end{cases}$$

$$x = |z| \cos \varphi, \quad y = |z| \sin \varphi.$$

Skaičiaus $z = x + iy$ jungtinis skaičius yra $\overline{z} = x - iy$.

$$|z| = |\overline{z}|, \quad \arg \overline{z} = -\arg z.$$

Išimtis. Jeigu $z = x < 0$, tai $\arg \overline{z} = \arg z = \pi$.

Trigonometrinė kompleksinio skaičiaus forma $z = |z|(\cos \varphi + i \sin \varphi)$.

Kompleksiniai skaičiai $z_1 = x_1 + iy_1$ ir $z_2 = x_2 + iy_2$ yra lygūs tada ir tik tada, kai $x_1 = x_2$ ir $y_1 = y_2$.

Jeigu kompleksiniai skaičiai užrašyti trigonometrine forma, tai $z_1 = z_2$ jeigu

$$|z_1| = |z_2|, \quad \text{ir} \quad \text{Arg}\, z_1 = \text{Arg}\, z_2 + 2\pi k.$$

Pavyzdžiai.

1. Rasti kompleksinio skaičiaus argumentą.

a) $\arg 1 = 0, \quad \text{Arg}\, 1 = 2\pi k;$

b) $\arg i = \dfrac{\pi}{2}, \quad \text{Arg}\, i = \dfrac{\pi}{2} + 2\pi k = \dfrac{4k+1}{2}\pi;$

c) $\arg(1 - i) = -\dfrac{\pi}{4}, \quad \text{Arg}\,(1 - i) = -\dfrac{\pi}{4} + 2\pi k;$

d) $\arg\left(-\dfrac{1}{2}\right) = \pi, \quad \text{Arg}\left(-\dfrac{1}{2}\right) = (2k+1)\pi.$

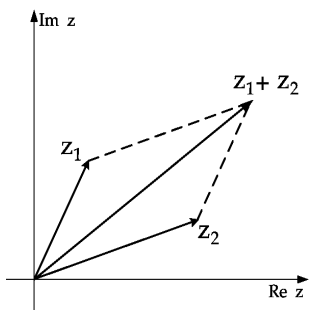
2. Rasti z, kuriems
a) $z = \overline{z}$; b) $z = |z|$.
Sprendimas.
a) $z = x + iy$, $\overline{z} = x - iy$, tuomet $y = -y$ ir $y = 0$, t. y. $z = x \in \mathbb{R}$.
b) $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$, $x + iy = \sqrt{x^2 + y^2}$, t. y. $y = 0$ ir $z = x \in \mathbb{R}$.

3. Rasti $|z|$ ir $\arg z$, jeigu

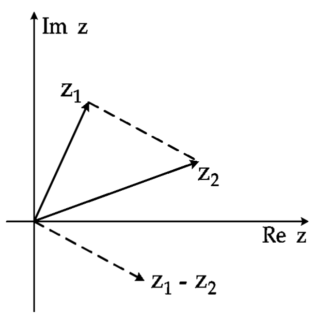
a) $z = -1$, b) $z = -2 + 2i$, c) $z = -2 - 2i$, d) $z = -3i$,
e) $z = 5i$, f) $z = 3 + 4i$, g) $z = 3 - 4i$.

4. Rasti ir pavaizduoti z, jeigu
a) $\mathrm{Re}\, z = \mathrm{Im}\, z$, b) $\mathrm{Re}\, z > 0$, c) $\arg z = \dfrac{\pi}{4}$,
d) $0 \leqslant \arg z \leqslant \dfrac{\pi}{2}$, e) $|z| = 1$, f) $|z| < 1$.

Veiksmai su kompleksiniais skaičiais



Suma
Jeigu $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, tai
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
Savybės:
1) $z_1 + z_2 = z_2 + z_1$,
2) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.



Skirtumas
Jeigu $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, tai
$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.
$|z_1 - z_2|$ - atstumas tarp taškų, vaizduojančių šiuos skaičius kompleksinėje plokštumoje.

Sandauga

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1), \quad z \cdot \overline{z} = x^2 + y^2.$$

Trigonometrine forma:

$$z_1 = |z_1|(\cos \text{Arg}\, z_1 + i \sin \text{Arg}\, z_1),$$

$$z_2 = |z_2|(\cos \text{Arg}\, z_2 + i \sin \text{Arg}\, z_2),$$

$$z_1 \cdot z_2 = |z_1||z_2| \left(\cos(\text{Arg}\, z_1 + \text{Arg}\, z_2) + i \sin(\text{Arg}\, z_1 + \text{Arg}\, z_2)\right).$$

Savybės:

1. $|z_1\, z_2| = |z_1|\, |z_2|, \quad \text{Arg}\,(z_1\, z_2) = \text{Arg}\, z_1\, \text{Arg}\, z_2,$
2. $z_1\, z_2 = z_2\, z_1,$
3. $(z_1\, z_2)\, z_3 = z_1\, (z_2\, z_3),$
4. $z_1\, (z_2 + z_3) = z_1\, z_2 + z_1\, z_3.$

Dalyba

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1\, x_2 + y_1\, y_2}{x_2^2 + y_2^2} + i \frac{x_2\, y_1 - x_1\, y_2}{x_2^2 + y_2^2}$$

Trigonometrine forma:

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \left(\cos(\text{Arg}\, z_1 - \text{Arg}\, z_2) + i \sin(\text{Arg}\, z_1 - \text{Arg}\, z_2)\right).$$

Savybės:

1. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$
2. $\text{Arg} \frac{z_1}{z_2} = \text{Arg}\, z_1 - \text{Arg}\, z_2.$

Pavyzdžiai

5. Rasti $\frac{z_1\, z_2}{z_3}$, jeigu $z_1 = 3 + 5i$, $z_2 = 2 + 3i$, $z_3 = 1 + 2i$.

Sprendimas.

$$\frac{z_1\, z_2}{z_3} = \frac{(3+5i)(2+3i)}{1+2i} = \frac{-9+19i}{1+2i} = \frac{(-9+19i)(1-2i)}{(1+2i)(1-2i)} =$$

$$= \frac{29+37i}{5} = \frac{29}{35} + \frac{37}{5} i.$$

6. Užrašyti trigonometrine forma skaičių $z = 2\sqrt{3} - 2i$.
Sprendimas.

$$|z| = 4, \quad \sin\varphi = -\frac{1}{2}, \quad \cos\varphi = \frac{\sqrt{3}}{2}, \quad \varphi = -\frac{\pi}{6}, \quad z = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

7. Užrašyti trigonometrine forma skaičius 1, i, -1, -i.

8. $z_1 = 1 + i$, $z_2 = \sqrt{3} + i$, $z_3 = 1 + \sqrt{3}i$. Užrašyti trigonometrine forma ir apskaičiuoti $\frac{z_1}{z_2\, z_3}$.
Sprendimas.

$$|z_1| = \sqrt{2}, \quad \arg z_1 = \frac{\pi}{4}, \quad z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right);$$
$$|z_2| = 2, \quad \arg z_2 = \frac{\pi}{6}, \quad z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right);$$
$$|z_3| = 2, \quad \arg z_1 = \frac{\pi}{3}, \quad z_3 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.\right).$$

$$z_2\, z_3 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right),$$
$$\frac{z_1}{z_2\, z_3} = \frac{\sqrt{2}}{4}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \frac{1}{4} - \frac{1}{4}i.$$

Kėlimas laipsniu

$$z^n = |z|^n\left(\cos n\varphi + i\sin n\varphi\right).$$

Šaknies traukimas

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right), \quad k = 0,\, 1,\, \ldots, n-1.$$

Pavyzdžiai

9. Apskaičiuoti $\sqrt[3]{z}$, jeigu $z = \sqrt{3} + i$.
Sprendimas

$$|z| = 2, \quad \varphi = \frac{\pi}{6}, \quad z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),$$

$$\alpha_k = \sqrt[3]{2}\left(\cos\frac{\pi/6 + 2\pi k}{3} + i\sin\frac{\pi/6 + 2\pi k}{3}\right), \quad k = 0,\ 1,\ 2;$$

$$\begin{aligned}
\alpha_0 &= \sqrt[3]{2}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right),\\
\alpha_1 &= \sqrt[3]{2}\left(\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right),\\
\alpha_2 &= \sqrt[3]{2}\left(\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right).
\end{aligned}$$

10. Apskaičiuoti

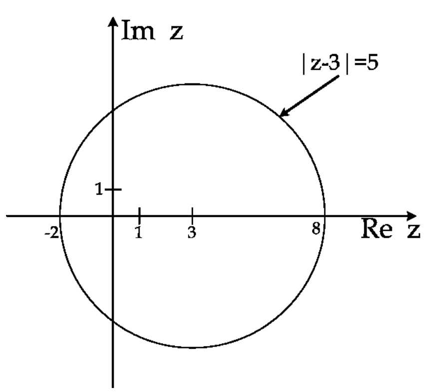
a) $(2+i)^0$, b) i^{-1}, c) $(2+i)^{-2}$, d) $\sqrt[4]{1}$;

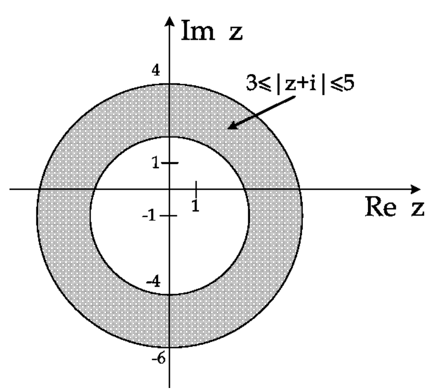
e) $\sqrt[3]{-8}$, f) $1^{2/3}$, g) $(i+1)^{10}$, h) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{24}$;

i) $\sqrt[3]{1}$, j) $\sqrt[4]{-1}$, k) $\sqrt[3]{i}$, l) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{10}$;

m) $\sqrt[4]{-i}$, n) $\sqrt[3]{8}$, o) $\sqrt{1+i}$, p) $(\sqrt{2} + \sqrt{2}i)^{25}$.

11. Pavaizduoti $|z - 3 = 5|$ ir $3 \leqslant |z - i| \leqslant 5$.





12. Rasti natūralųjį n, tokį kad $(1+i)^n = (1-i)^n$.

Sprendimas. Užrašome skaičius $z_1 = 1+i$ ir $z_2 = 1-i$ trigonometrine forma

$$z_1 = 1+i, \quad z_1 = \sqrt{2}, \quad \varphi_1 = \frac{\pi}{4}, \qquad z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right);$$
$$z_2 = 1-i, \quad z_2 = \sqrt{2}, \quad \varphi_2 = -\frac{\pi}{4}, \quad z_2 = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).$$

Tada

$$(\sqrt{2})^n\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right) = (\sqrt{2})^2\left(\cos\left(-\frac{\pi n}{4}\right) + i\sin\left(-\frac{\pi n}{4}\right)\right),$$

$$\cos\frac{\pi n}{4} = \cos\left(-\frac{\pi n}{4}\right), \quad \sin\frac{\pi n}{4} = \sin\left(-\frac{\pi n}{4}\right),$$

$$2\sin\frac{\pi n}{4} = 0, \quad \frac{\pi n}{4} = \pi k, \quad n = 4k, \quad k \in \mathbb{N}.$$

13. Išspręsti lygtį $z^2 + \overline{z} = 0$.

Sprendimas. Jei $z = x + iy$, tai

$$(x+iy)^2+x-iy = 0, \quad x^2+2xyi-y^2+x-yi = 0, \quad (x^2-y^2+x)+(2x-1)yi = 0,$$

Sudarome lygčių sistemą ir išsprendžiame ją:

$$\begin{cases} x^2 - y^2 + x = 0, \\ (2x-1)y = 0; \end{cases}$$

$$z_1 = 0, \quad z_2 = -1, \quad z_3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_4 = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

14. Išspręsti lygtį $\overline{z} = -4z$.

15. Pavaizduoti sritį Re $(iz + 2 - 3i) > 5$.

16. Pavaizduoti kreivę $|z - i| = |z + 2|$.